



Research Article

On L -Fuzzy Fixed Point Results and Related Applications

Mohammed Shehu Shagari, Musa Balarabe and Kabir Lawal Yusuf
Department of Mathematics, Faculty of Physical Sciences Ahmadu Bello
University, Zaria, Nigeria
Corresponding author: shagaris@gmail.com

Keywords and Phrases:

L -Fuzzy Set,
 L -Fuzzy set-valued map,
Fixed point,
 η_* - admissible mapping,
 $\eta_* - \psi$ -contractive mapping
Metric space

ABSTRACT

Amidst several efforts in advancing fuzzy mathematics, a lot of attentions have been paid to investigate novel L -fuzzy analogues of the conventional fixed point results and their various applications. In this direction, a few new L -fuzzy fixed point theorems for L -fuzzy mappings are put forward in this paper. From application consideration, corresponding L -fuzzy fixed point results in ordered metric spaces are highlighted and analyzed. A nontrivial example is provided to validate the hypotheses of our obtained results. As some consequences, we note that the notions initiated herein complement, harmonize, and extend more than a handful of some recently proposed results in the related literature.

Corresponding author: Mohammed Shehu Shagari, Email: shagaris@gmail.com
Ahmadu Bello University, Zaria, Nigeria

INTRODUCTION

Fixed point theory is the epicenter of modern mathematical analysis with enormous applications in the study of various significant nonlinear phenomena. In metric fixed point theory, the contractive conditions on a given mapping play an important role in solving fixed point problems. The Banach contraction mapping principle (see Banach, 1922) is one of the first well-known results in metric fixed point theory. This highly

celebrated result, which is an essential tool in many areas of mathematics and related disciplines, appeared in an explicit form in 1922 in Banach thesis, where it was mainly used to examine the existence of solution to an integral equation. Due to its importance and simplicity, many authors have come up with diverse extensions of the contraction mapping principle (Hussain *et al.* 2013; Kutbi *et al.*, 2013; Samet *et al.*, 2012). Along this line, (Samet *et al.*, 2012) initiated the concepts of $\eta - \psi$ contractive and η -admissible

mappings and presented several fixed point results for such mappings. In particular, the Banach fixed point result was obtained as a consequence of the main ideas in (Samet *et al.*, 2012).

Denote by Ξ , the class of functions $\psi : \mathbb{R}_+ = [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{m=1}^{\infty} \psi^m(t) < \infty$ for all $t > 0$, where $\psi^n(t)$ is the n th iterate of ψ . This class of functions Ξ are known in the literature as altering distance functions. The following result is well-familiar.

Lemma 1.1. *If $\psi \in \Xi$, then the following hold:*

- (i) $(\psi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$ for all $t \in (0, \infty)$.
- (ii) $\psi(t) < t$ for all $t > 0$.
- (iii) $\psi(t) = 0$ if and only if $t = 0$.

(Samet *et al.*, 2012) defined the concept of η -admissible mappings in the following manner.

Definition 1.2. (Samet *et al.*, 2012) *Let f be a self-mapping on a nonempty set X and $\eta : X^2 \rightarrow [0, \infty)$ be a function. We say that f is an η -admissible mapping, if*

$$x, y \in X \ \eta(x, y) \geq 1 \text{ implies } \eta(fx, fy) \geq 1.$$

Let (X, d) be a metric space and $CB(X)$ denotes the set of all nonempty closed and bounded subsets of X . For $A, B \in CB(X)$, the function $H : CB(X) \times CB(X) \rightarrow \mathbb{R}_+ = [0, \infty)$ defined by

$$H(A, B) = \max_{a \in A, b \in B} \{ \sup d(a, B), \sup d(b, A) \},$$

is called the Pompeiu-Hausdorff metric on $CB(X)$, where $d(x, A) = \inf_{a \in A} \{ d(x, a) : a \in A \}$.

Remark 1.3. *It is easy to see that for all $a, c \in X$ and $B, C \in CB(X)$, the following are true:*

- (i) $d(a, B) \leq d(a, c) + d(c, B)$.
- (ii) $d(a, B) \leq d(a, c) + H(C, B)$.

Following (Samet *et al.*, 2012), Asl *et al.* (2012) introduced the notions of η_* - ψ - contractive multi-functions and η_* -admissible and proved the corresponding fixed point theorems.

Definition 1.4. *Let (X, d) be a metric space and $F : X \rightarrow CB(X)$ be a multivalued mapping. We say that F is an $\eta_* - \psi$ -contractive multifunction, if there exist two functions $\eta : X^2 \rightarrow \mathbb{R}_+$ and $\psi \in \Xi$ such that for all $x, y \in X$,*

$$\eta_*(Fx, Fy)H(Fx, Fy) \leq \psi(d(x, y)),$$

where $\eta_*(A, B) = \inf \{ \eta(a, b) : a \in A, b \in B \}$.

Definition 1.5. [2] *Let (X, d) be a metric space, $F : X \rightarrow CB(X)$ be a multivalued mapping and $\eta : X^2 \rightarrow \mathbb{R}_+$ be a function. We say that F is an η_* -admissible whenever $\eta(x, y) \geq 1$ implies $\eta_*(Fx, Fy) \geq 1$.*

Not long ago, (Hussain *et al.*, 2013a) modified the ideas of η_* -admissible and $\eta_* - \psi$ contractive mappings as follows.

Definition 1.6. (Hussain *et al.*, 2013a) *Let $F : X \rightarrow CB(X)$ be a multivalued mapping, $\eta, \theta : X^2 \rightarrow \mathbb{R}_+$ be two functions, where θ is bounded. We say that F is an η_* -admissible mapping with respect to θ if for all $x, y \in X$,*

$$\eta(x, y) \geq \theta(x, y) \text{ implies } \eta_*(Fx, Fy) \geq \theta_*(Fx, Fy),$$

where $\theta_*(A, B) = \sup \{ \theta(a, b) : a \in A, b \in B \}$.

Note that if $\theta(x, y) = 1$ for all $x, y \in X$, then Definition 1.6 reduces to Definition 1.5. For more recent results on $\eta - \psi$ -contractive and η -admissible mappings with related fixed point theorems, the reader may consult (Ali & Kamran, 2013; Hussain *et al.*, 2013b; Kutbi *et al.*, 2013; Mohammed, 2021).

On the other hand, from the inception of creation, man has always been making enormous efforts in comprehending nature and then developing a strong link between life and its requirements.

These efforts comprise of three stages, viz. comprehending the surrounding environment, acknowledgement of novelty, and planning for the future. In these struggles, many issues like linguistic interpretation, characterization of interrelated phenomena into proper categories, use of restricted ideas, uncertainty in data analysis, and a host of others, affect the accuracy of results. The above mentioned problems inherent with everyday life can be overcome by availing the notions of fuzzy sets due to their flexibility in nature as compared to crisp sets. After the introduction of fuzzy sets by (Zadeh, 1965), various areas of mathematics, social sciences and engineering witness tremendous revolutions. Specifically, fuzzy set is characterized by a membership function which assigns to each of its elements a grade of membership ranging between zero and one. Meanwhile, the basic notions of fuzzy sets have been improved and applied in different directions. Along this direction, we can refer (Rasham, *et al.*, 2021a, 2021b, 2021c, 2021d; Mohammed *et al.*, 2020) In 1981, (Heilpern, 1981) used the idea of fuzzy set to introduce a class of fuzzy set-valued maps and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of fixed point theorems due to (Nadler, 1965) and (Banach, 1922). Afterwards, several authors have studied the existence of fixed points of fuzzy set-valued maps, for example, see (Azam & Beg, 2009; Mohammed & Azam, 2019; Mohammed *et al.*, 2021; Shagari & Azam, 2020). A very interesting generalization of fuzzy sets by replacing the interval $[0,1]$ of range set by a complete distributive lattice was initiated by (Goguen, 1967) and called L -fuzzy sets. Not long ago, (Rashid *et al.*, 2014a) brought up the notion of L -fuzzy mappings and presented a common fixed point theorem through the idea of β_{FL} -admissible pair of L -fuzzy mappings. As an improvement of the concept of Hausdorff distance and d_∞ -metric for fuzzy sets, (Rashid *et al.*, 2014b)

launched the ideas of $D_{\alpha L}$ and d^∞_L distances for L -fuzzy sets and refined some known fixed point theorems for fuzzy and crisp set-valued mappings.

Definition 1.7. A relation on a nonempty set L is called a partial order if it is

- (i) Reflexive
- (ii) Antisymmetric
- (iii) Transitive.

A set L together with a partial ordering is called a partially ordered set (poset, for short) and is denoted by (L, \preceq_L) . Recall that partial orderings are used to give an order to sets that may not have a natural one.

Definition 1.8. Let L be a nonempty set and (L, \preceq) be a partially ordered set. Then any two elements $x, y \in L$ are said to be comparable if either $x \preceq y$ or $y \preceq x$.

Definition 1.9. A partially ordered set (L, \preceq_L) is called:

- (i) a lattice, if $x \vee y \in L, x \wedge y \in L$ for any $x, y \in L$;
- (ii) a complete lattice, if $\forall \nabla \in L, \wedge \nabla \in L$ for any $\nabla \subseteq L$;
- (iii) distributive lattice if $x \vee (y \wedge \zeta) = (x \vee y) \wedge (x \vee \zeta), x \wedge (y \vee \zeta) = (x \wedge y) \vee (x \wedge \zeta)$, for any $x, y, \zeta \in L$.

A partially ordered set L is called a complete lattice if for every doubleton $\{x, y\}$ in L , either $\sup\{x, y\} = x \vee y$ or $\inf\{x, y\} = x \wedge y$ exists.

Definition 1.10. (Goguen, 1967) An L -fuzzy set ∇ on a nonempty set X is a function with domain X and whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L , respectively.

Denote the class of all L -fuzzy sets on a nonempty set X by L^X (to mean a function: $X \rightarrow L$).

Definition 1.11. (Goguen, 1967) The α_L -level set of an L -fuzzy set ∇ is denoted by $[\nabla]_{\alpha L}$ and is defined as follows:

$$[\nabla]_{\alpha_L} = \begin{cases} \overline{\{x \in X : 0_L \preceq_L \nabla(x)\}}, & \text{if } \alpha_L = 0 \\ \{x \in X : \alpha_L \preceq_L \nabla(x)\}, & \text{if } \alpha_L \in L \setminus \{0_L\} \end{cases}$$

Definition 1.12. (Rashid et al., 2014a; Rashid et al., 2014b) Let X be an arbitrary nonempty set and Y a metric space. A mapping $T : X \rightarrow L^Y$ is called an L -fuzzy mapping. The function value $T(x)(y)$ is called the degree of membership of y in $T(x)$. For an L -fuzzy mappings $T : X \rightarrow L^Y$, a point $u \in X$ is called an L -fuzzy fixed point of T if there exists $\alpha_L \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L}$.

Consistent with (Rashid et al., 2014a; Rashid et al., 2014b), we define some needed auxiliary concepts in the setting of metric space as follows. Let (X, d) be a metric space and take $\alpha_L \in L \setminus \{0_L\}$ such that $[\nabla]_{\alpha_L}, [\Delta]_{\alpha_L} \in CB(X)$. Then, define

$$p\alpha_L(\nabla, \Delta) = \inf \{ d(x, y) : x \in [\nabla]_{\alpha_L}, y \in [\Delta]_{\alpha_L} \} .$$

$$D\alpha_L(\nabla, \Delta) = H([\nabla]_{\alpha_L}, [\Delta]_{\alpha_L}). \quad p(\nabla, \Delta) = \sup \alpha_L p\alpha_L(\nabla, \Delta). \quad d_L^\infty(\nabla, \Delta) = \sup \alpha_L D\alpha_L(\nabla, \Delta).$$

The following lemmas will be needed in the sequel.

Lemma 1.13. (Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space (X, d) and $0 < h \in \mathbb{R}$. Then, for each $b \in B$, there exists $a \in A$ such that $d(a, b) \leq H(A, B) + h$.

Lemma 1.14. (Ali & Kamran, 2013) Let (X, d) be a metric space and B be a nonempty closed subset of X and $q > 1$. Then, for every $x \in X$ with $d(x, B) > 0$, there exists $b \in B$ such that $d(x, b) \leq qd(x, B)$.

The aim of this note is to initiate a new notion of L -fuzzy set-valued maps under the name η_* - (ψ, K) -weak L -fuzzy contractions and investigate sufficient conditions for existence of L -fuzzy fixed points for such contractions in the framework of complete metric spaces. As some consequences of the ideas put forward herein, a few corresponding fixed point results in ordered metric space are pointed out and discussed. A nontrivial example is

presented to authenticate the hypotheses of our obtained results.

RESULTS

Progressing along the concepts established in (Asl et al., 2012; Berinde, 2004; Hussain et al., 2013a; Hussain et al., 2014; Samet et al., 2012), we launch this section with some preliminary notions which are relevant in investigating our main L -fuzzy fixed point results. Throughout this paper, by $L_{CB(X)}$, we mean

$$L_{CB(X)} = \{A \in L^X : [A]_{\alpha_L} \in CB(X), \text{ for each } \alpha_L \in L \setminus \{0_L\}\}.$$

Definition 2.1. Let (X, d) be a metric space, $T : X \rightarrow L_{CB(X)}$ be an L -fuzzy set valued map. Then T is called an $\eta_* - \psi$ -contractive L -fuzzy set-valued map, if there exist three mappings $\alpha_L : X \rightarrow L \setminus \{0_L\}$, $\eta : X^2 \rightarrow \mathbb{R}^+$ and $\psi \in \Xi$ such that for all $x, y \in X$,

$$\eta_*([\nabla]_{\alpha_L}(x), [\nabla]_{\alpha_L}(y))H([\nabla]_{\alpha_L}(x), [\nabla]_{\alpha_L}(y)) \leq \psi(d(x, y)).$$

Definition 2.2. Let (X, d) be a metric space, $T : X \rightarrow L_{CB(X)}$ be an L -fuzzy set valued map. Then T is said to be η_* -admissible, if there exist two mappings

$$\eta : X^2 \rightarrow \mathbb{R}^+, \quad \alpha_L : X \rightarrow L \setminus \{0_L\}$$

such that for all $x, y \in X$,

$$\eta(x, y) \geq 1 \text{ implies } \eta_*([\nabla]_{\alpha_L}(x), [\nabla]_{\alpha_L}(y)) \geq 1.$$

Definition 2.3. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, d) is called a trajectory of an L -fuzzy set-valued map $T : X \rightarrow L^X$, starting at x_1 , if $x_{n+1} \in [Tx_n]_{\alpha_L(x_n)}$ for each $\alpha_L(x_n) \in L \setminus \{0_L\}$ and $n \in \mathbb{N}$.

We denote the family of all trajectories of an L fuzzy set-valued map T at $x \in X$

by $T_r(x)$.

Definition 2.4. (Shukla et al., 2017) Let (X, d) be a metric space and $\eta : X^2 \rightarrow \mathbb{R}^+$ be a mapping. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called an η -sequence if $\eta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$

We denote the family of all η -sequence in X by $E_{X\eta}$. For an L -fuzzy set-valued map $T : X \rightarrow L_{CB(X)}$, we denote the η -graph of T by $G_\eta(T)$ and is defined as:

$$G_\eta(T) = \{(x_n, x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_n \in E_{X\eta} \cap T_{r(x_1)}\}.$$

The mapping T is called an η -closed L -fuzzy set-valued map if $G_\eta(T)$ is a closed subset of X^2 . Note that every closed mapping is a particular kind of η -closed mapping (cf. (Shukla et al., 2017)).

Definition 2.5. (Shukla et al., 2017) Let (X, d) be a metric space and $\eta : X^2 \rightarrow \mathbb{R}^+$ be a mapping. Then (X, d) is said to be η -complete if every Cauchy sequence in $E_{X\eta}$ is convergent in X^2 .

Note that every complete metric space is η -complete, but the converse needs not be true (cf. (Shukla et al., 2017)).

Throughout this paper, for all $x, y \in X$, we define $\Omega_L T(x, y)$ as :

$$\Omega_L T(x, y) = \max \left\{ d(x, y), d(x, [Tx]_{\alpha_L(x)}), d(y, [Ty]_{\alpha_L(y)}), d(x, [Ty]_{\alpha_L(y)}), d(y, [Tx]_{\alpha_L(x)}), \frac{d(x, [Tx]_{\alpha_L(x)})d(y, [Ty]_{\alpha_L(y)})}{1 + d(x, y)} \right\}$$

Definition 2.6. Let (X, d) be a metric space and $T : X \rightarrow L^X$ be an L -fuzzy setvalued map. Then T is called (ψ, K) -weak L -fuzzy contraction, if there exist $\psi \in \Xi$, $K \geq 0$ and a mapping $\alpha_L : X \rightarrow L \setminus \{0_L\}$ such that for all $x, y \in X$,

$$H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \psi(\Omega_L T(x, y)) + Kd(y, [Tx]_{\alpha_L(x)}). \tag{2.1}$$

Following (Berinde, 2004), we give the next definition which is an extension of Definition 2.6 and an L -fuzzy extension of the main idea in (Berinde, 2004).

Definition 2.7. Let (X, d) be a metric space, $T : X \rightarrow L^X$ be an L -fuzzy set-valued map and $\eta : X^2 \rightarrow \mathbb{R}^+$ be a mapping. Then T is called an η^* - (ψ, K) -weak L -fuzzy contraction, if there exist $\psi \in \Xi$, $K \geq 0$ and a mapping $\alpha_L : X \rightarrow L \setminus \{0_L\}$ such that for all $x, y \in X$ and $\eta^*([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \geq 1$,

$$H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \psi(\Omega_L T(x, y)) + Kd(y, [Tx]_{\alpha_L(x)}). \tag{2.2}$$

Remark 2.8. Consistent with Berinde (Berinde, 2004), we note that due to the symmetry of the distance function, the contraction condition (2.1) implicitly includes the following dual one:

$$H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \psi(\Omega_L T(x, y)) + Kd(x, [Ty]_{\alpha_L(y)}). \tag{2.3}$$

for all $x, y \in X$. Therefore, in order to verify (ψ, K) -weak L -fuzzy contraction of T , it is necessary to check both the conditions (2.2) and (2.3). However, for η^* - (ψ, K) -weak L -fuzzy contraction, the inequality (2.2) is valid only for those $x, y \in X$ with $\alpha_L(x), \alpha_L(y) \in L \setminus \{0_L\}$ for which $\eta^*([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \geq 1$. The advantage here is that it is not necessary to verify condition (2.2) for all $x, y \in X$.

Now, we state and prove our main L -fuzzy fixed point result as follows.

Theorem 2.9. Let (X, d) be an η -complete metric space and $T : X \rightarrow L^X$ be an η^* - (ψ, K) -weak L -fuzzy contraction. Suppose further that the following conditions are satisfied:

(C₁) for each $x \in X$, there exists $\alpha_L(x) \in L \setminus \{0_L\}$ such that $[Tx]_{\alpha_L(x)}$ is a nonempty closed and bounded subset of X ;

(C₂) T is η^* -admissible;

(C₃) there exists $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ with $\eta(x_0, x_1) \geq 1$;

(C₄) for a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then, there exists $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$.

Proof. By Condition (C₃), there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ such that $\eta(x_0, x_1) \geq 1$. If $x_0 = x_1$, the proof is finished. We presume that $x_0 \neq x_1$. If there exists $x_1 \in [Tx_1]_{\alpha_L(x_1)}$ for some $\alpha_L(x_1) \in L \setminus \{0_L\}$, then x_1 is an L -fuzzy fixed point of T . So suppose that $x_1 \notin [Tx_1]_{\alpha_L(x_1)}$. Then, given that T is η^* -admissible and $\eta(x_0, x_1) \geq 1$, we have $\eta^*([Tx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}) \geq 1$. It follows that

$$d(x_1, [Tx_1]_{\alpha_L(x_1)}) \leq H([Tx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}) \tag{2.4} \leq \psi(\Omega_T(x_0, x_1)) + Kd(x_1, [Tx_0]_{\alpha_L(x_0)}),$$

where

$$\begin{aligned} \Omega_L T(x_0, x_1) &= \max \left\{ d(x_0, x_1), d(x_0, [Tx_0]_{\alpha_L(x_0)}), d(x_1, [Tx_1]_{\alpha_L(x_1)}), \right. \\ &\quad \left. d(x_0, [Tx_1]_{\alpha_L(x_1)})d(x_1, [Tx_0]_{\alpha_L(x_0)}), \frac{d(x_0, [Tx_0]_{\alpha_L(x_0)})d(x_1, [Tx_1]_{\alpha_L(x_1)})}{1 + d(x_0, x_1)} \right\} \\ &\leq \left\{ d(x_0, x_1), d(x_0, x_1), d(x_1, [Tx_1]_{\alpha_L(x_1)}), \frac{d(x_0, x_1)d(x_1, [Tx_1]_{\alpha_L(x_1)})}{1 + d(x_0, x_1)} \right\} \\ &\leq \max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha_L(x_1)})\}. \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha_L(x_1)})\} = d(x_1, [Tx_1]_{\alpha_L(x_1)})$, then by Lemma 1.1, it follows from (2.4) that

$$d(x_1, [Tx_1]_{\alpha_L(x_1)}) \leq \psi(d(x_1, [Tx_1]_{\alpha_L(x_1)})) < d(x_1, [Tx_1]_{\alpha_L(x_1)}),$$

a contradiction. Hence,

$$\max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha_L(x_1)})\} = d(x_0, x_1).$$

Consequently,

$$d(x_1, [Tx_1]_{\alpha_L(x_1)}) \leq \psi(d(x_0, x_1)). \tag{2.5}$$

For any $y > 1$, by Lemma 1.14, there exists $x_2 \in [Tx_1]_{\alpha_L(x_1)}$ such that

$$d(x_1, x_2) \leq yd(x_1, [Tx_1]_{\alpha_L(x_1)}) \leq y\psi(d(x_0, x_1)).$$

Clearly, if $x_1 = x_2$, then T has at least one L -fuzzy fixed point in X and the proof is complete.

Suppose that $x_1 \neq x_2$ so that $d(x_1, x_2) > 0$. Take

$$y_1 = \frac{\psi(\psi(d(x_0, x_1)))}{\psi(d(x_1, x_2))}. \text{ Then,}$$

$y_1 > 1$. Since $x_1 \in [Tx_0]_{\alpha_L(x_0)}, x_2 \in [Tx_1]_{\alpha_L(x_1)}$ and $\eta^*([Tx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}) \geq 1$, we get $\eta(x_1, x_2) \geq 1$. And, η^* -admissibility of T implies that $\eta^*([Tx_1]_{\alpha_L(x_1)}, [Tx_2]_{\alpha_L(x_2)}) \geq 1$.

If $x_2 \in [Tx_2]_{\alpha_L(x_2)}$ for some $\alpha_L(x_2) \in L \setminus \{0_L\}$, then x_2 is an L -fuzzy fixed point of

T . So, let $x_2 \notin [Tx_2]_{\alpha_L(x_2)}$, then

$$d(x_2, [Tx_2]_{\alpha_L(x_2)}) \leq H([Tx_1]_{\alpha_L(x_1)}, [Tx_2]_{\alpha_L(x_2)}) \tag{2.6} \leq \psi(\Omega_T(x_1, x_2)) + Kd(x_2, [Tx_1]_{\alpha_L(x_1)}),$$

where

$$\begin{aligned} \Omega_L T(x_1, x_2) &= \max \left\{ d(x_1, x_2), d(x_1, [Tx_1]_{\alpha_L(x_1)}), d(x_2, [Tx_2]_{\alpha_L(x_2)}), \right. \\ &\quad \left. d(x_1, [Tx_2]_{\alpha_L(x_2)})d(x_2, [Tx_1]_{\alpha_L(x_1)}), \frac{d(x_1, [Tx_1]_{\alpha_L(x_1)})d(x_2, [Tx_2]_{\alpha_L(x_2)})}{1 + d(x_1, x_2)} \right\} \\ &\leq \left\{ d(x_1, x_2), d(x_1, x_2), d(x_2, [Tx_2]_{\alpha_L(x_2)}), \frac{d(x_1, x_2)d(x_2, [Tx_2]_{\alpha_L(x_2)})}{1 + d(x_1, x_2)} \right\} \\ &\leq \max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha_L(x_2)})\}. \end{aligned}$$

Assume that $\max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha_L(x_2)})\} = d(x_2, [Tx_2]_{\alpha_L(x_2)})$, then by Lemma 0.1, (0.6) yields

$$d(x_2, [Tx_2]_{\alpha_L(x_2)}) \leq \psi(d(x_2, [Tx_2]_{\alpha_L(x_2)})) < d(x_2, [Tx_2]_{\alpha_L(x_2)}),$$

a contradiction. Therefore, $\max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha_L(x_2)})\} = d(x_1, x_2)$. It follows that

$$d(x_2, [Tx_2]_{\alpha_L(x_2)}) \leq \psi(d(x_1, x_2)). \tag{0.7}$$

Since $\omega_1 > 1$, then by Lemma 0.14, there exists $x_3 \in [Tx_2]_{\alpha_L(x_2)}$ such that

$$\begin{aligned} d(x_2, x_3) &\leq \omega_1 d(x_2, [Tx_2]_{\alpha_L(x_2)}) \leq \omega_1 \psi(d(x_1, x_2)) \\ &= \psi(\omega \psi(d(x_0, x_1))). \end{aligned}$$

By continuing in this fashion, we construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in [Tx_{n-1}]_{\alpha_L(x_{n-1})}$, $x_n \neq x_{n-1}$, $\eta(x_n, x_{n-1}) \geq 1$ for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n-1}) \leq \psi^{n-1}(\omega \psi(d(x_0, x_1))). \tag{0.8}$$

Next, we shall show that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . For this, let $m, n \in \mathbb{N}$ with $m > n$, then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(\omega \psi(d(x_0, x_1))). \tag{0.9}$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, it follows from (0.9) that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . Therefore, $\{x_n\}_{n \in \mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ is a Cauchy sequence. The η -completeness of this space implies that there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$.

To show that u is an L -fuzzy fixed point of T , assume contrary that $u \notin [Tu]_{\alpha_L(u)}$ with $d(u, [Tu]_{\alpha_L(u)}) > 0$. Now, since $\{x_n\}_{n \in \mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ and $x_n \rightarrow u$ as $n \rightarrow \infty$,

it follows from (C_4) that $\eta(x_n, u) \geq 1$ for each $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}
 d(u, [Tu]_{\alpha_L(u)}) &\leq d(u, x_{n+1}) + d(x_{n+1}, [Tu]_{\alpha_L(u)}) \\
 &\leq d(u, x_{n+1}) + H([Tx_n]_{\alpha_L(x_n)}, [Tu]_{\alpha_L(u)}) \\
 &\leq \psi(\Omega_T(x_n, u)) + Kd(u, [Tx_n]_{\alpha_L(x_n)}) + d(x_{n+1}, u) \\
 &\leq \psi(\Omega_T(x_n, u)) + Kd(u, x_{n+1}) + d(x_{n+1}, u),
 \end{aligned} \tag{0.10}$$

where

$$\begin{aligned}
 \Omega_L T(x_n, u) &= \max \left\{ d(x_n, u), d(x_n, [Tx_n]_{\alpha_L(x_n)}), d(u, [Tu]_{\alpha_L(u)}), \right. \\
 &\quad \left. d(x_n, [Tu]_{\alpha_L(u)})d(u, [Tx_n]_{\alpha_L(x_n)}), \frac{d(x_n, [Tx_n]_{\alpha_L(x_n)})d(u, [Tu]_{\alpha_L(u)})}{1 + d(x_n, u)} \right\} \\
 &\leq \max \left\{ d(x_n, u), d(x_n, x_{n+1}), d(u, [Tu]_{\alpha_L(u)}), \right. \\
 &\quad \left. d(x_n, [Tu]_{\alpha_L(u)})d(u, x_{n+1}), \frac{d(x_n, x_{n+1})d(u, [Tu]_{\alpha_L(u)})}{1 + d(x_n, u)} \right\}.
 \end{aligned}$$

Taking limit in (0.10) as $n \rightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $\Omega_T(x_n, u) = d(u, [Tu]_{\alpha_L(u)})$ as $n \rightarrow \infty$ for all $n \geq n_0$. Hence, Lemma 0.1 implies that

$$d(u, [Tu]_{\alpha_L(u)}) \leq \psi(d(u, [Tu]_{\alpha_L(u)})) < d(u, [Tu]_{\alpha_L(u)}),$$

is a contradiction. Consequently, there exists $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$; that is, u is an L -fuzzy fixed point of T . \square

Corollary 0.24. *Let (X, d) be an η -complete metric space and $T : X \rightarrow L^X$ be an L -fuzzy set-valued map. Suppose further that the following conditions are satisfied:*

(C_1) *for each $x \in X$, there exists $\alpha_L(x) \in L \setminus \{0_L\}$ such that $[Tx]_{\alpha_L(x)} \in CB(X)$;*

(C_2) *there exists $\gamma \in (0, 1)$ and $K \geq 0$ such that for all $x, y \in X$,*

$$H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \gamma d(x, y) + Kd(y, [Tx]_{\alpha_L(x)});$$

(C_3) *T is η_* -admissible;*

(C_4) *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*

(C_5) *for a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exist $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$.

Proof. Take $\psi(t) = \gamma t$ for all $t \geq 0$ in Theorem 0.23. \square

The next result in association with d_L^∞ -metric for L -fuzzy sets extends the ideas presented in (Azam *et al.*, 2010; Heilpern, 1981; Park & Jeong, 1997). It is pertinent to note that L -fuzzy fixed-point results in the setting of d_∞ -metric is very useful in computing Hausdorff dimensions. These dimensions help us to understand the concepts of ε^∞ -space which is of great importance in higher energy physics (see, e.g. (ElNaschi, 2002; El-Naschi, 2000).

For all $x, y \in X$, we define $\Omega_L T(x, y, p)$ as:

$$\Omega_L T(x, y, p) = \max \left\{ d(x, y), p(x, T(x)), p(y, T(y)), p(x, T(y))p(y, T(x)), \frac{p(x, T(x))p(y, T(y))}{1 + d(x, y)} \right\}$$

Theorem 2.11. *Let (X, d) be an η -complete metric space and $T : X \rightarrow W(X)$ be an L -fuzzy set-valued map. Assume that the following conditions are satisfied:*

(C₁) *there exists $K \geq 0$ such that for all $x, y \in X$,*

$$d^\infty_L(Tx, Ty) \leq \psi(\Omega_L T(x, y, p)) + Kp(y, T(x)); \tag{2.11}$$

(C₂) *T is η^* -admissible;*

(C₃) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ with $\eta(x_0, x_1) \geq 1$;*

(C₄) *for a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, T has an L -fuzzy fixed point in X .

Proof. Let $x \in X$, by assumption, $[Tx]_{1L}$ is a nonempty closed and bounded subset of

X . Now, for all $x, y \in X$,

$$\begin{aligned} D_{1_L}(T(x), T(y)) &\leq d_L^\infty(T(x), T(y)) \\ &\leq \psi(\Omega_T(x, y, p)) + Kp(y, T(x)). \end{aligned}$$

Since $[Tx]_{1L} \subseteq [Tx]_{\alpha_L(x)} \in CB(X)$ for each $\alpha_L(x) \in L \setminus \{0_L\}$, therefore, $d(x, [Tx]_{\alpha_L(x)}) \leq d(x, [Tx]_{1L})$ for each $\alpha_L(x) \in L \setminus \{0_L\}$. This implies that $p(x, T(x)) \leq d(x, [Tx]_{1L})$. On similar arguments, $p(y, T(x)) \leq d(y, [Tx]_{1L})$. Hence,

$$\begin{aligned} H([Tx]_{1_L}, [Ty]_{1_L}) &\leq D_{1_L}(T(x), T(y)) \\ &\leq \psi(\Omega_L T(x, y, p)) + Kp(y, T(x)) \\ &\leq \psi(\Omega_L T(x, y)) + Kd(y, [Tx]_{1L}). \end{aligned}$$

Hence, Theorem 2.9 can be employed to find $u \in X$ and $\alpha_L(u) = 1_L$ such that $u \in [Tu]_{1L}$.

In the next result, we replace Condition (C₄) of Theorem 2.9 with η -closedness of the mapping T .

Theorem 2.12. *Let (X, d) be an η -complete metric space and $T : X \rightarrow L^X$ be an η^* - (ψ, K) -weak L -fuzzy contraction. Assume further that the following conditions are satisfied:*

(C₁) *for each $x \in X$, there exists $\alpha_L(x) \in (0, 1]$ such that $[Tx]_{\alpha_L(x)} \in CB(X)$;*

(C₂) *T is η^* -admissible;*

(C₃) *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*

(C₄) *T is an η -closed fuzzy set-valued map.*

Then, there exist $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$.

Proof. In line with the proof of Theorem 2.9, we have a sequence $\{x_n\}_{n \in \mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ such that $x_n \rightarrow u \in X$ as $n \rightarrow \infty$. To see that u is an L -fuzzy fixed point of T , first notice that since $\{x_n\}_{n \in \mathbb{N}} \in E_{X\eta} \cap$

$T_{r(x_0)}$, then by definition of $G_\eta(T)$, we get $(x_{n-1}, x_n) \in G_\eta(T)$ for all $n \in \mathbb{N}$. Given that T is an η -closed L -fuzzy set-valued map, the η -graph $G_\eta(T)$ is a closed subset of X^2 . Thus, letting $n \rightarrow \infty$ and noting that $x_n \rightarrow u$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} (x_{n-1}, x_n) \in G_\eta(T) \text{ implies } (u, u) \in G_\eta(T).$$

By definition of $G_\eta(T)$, we obtain some $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$.

In what follows, we provide an example to support the hypotheses of Theorem 2.9.

Example 2.13. Let $L = \{a, b, c, g, s, m, n, v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v, a \preceq_L g \preceq_L b \preceq_L v, s \preceq_L m \preceq_L v, g \preceq_L m \preceq_L v, n \preceq_L v$; and each elements of the doubletons $\{c, m\}, \{m, b\}, \{s, n\}, \{n, g\}$ are not comparable. It follows that (L, \preceq_L) is a complete distributive lattice. Let $X = [0, \infty)$ and $d : X^2 \rightarrow \mathbb{R}^+$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \max\{x, y\}, & \text{if } x \neq y. \end{cases}$$

Then, (X, d) is an η -complete metric space. Let $\alpha_L : X \rightarrow L \setminus \{0_L\}$ be a mapping. For each $x \in X$, consider an L -fuzzy set-valued map $T(x) : X \rightarrow L$ defined as follows: if $x = 0$, then

$$T(x)(t) = \begin{cases} v, & \text{if } t = 0 \\ a, & \text{if } t \neq 0, \end{cases}$$

if $x \in (0, 4]$, then

$$T(x)(t) = \begin{cases} v, & \text{if } 0 \leq t < \frac{x}{50} \\ b, & \text{if } \frac{x}{50} \leq t < \frac{x}{17} \\ c, & \text{if } \frac{x}{17} \leq t < \frac{x}{3} \\ a, & \text{if } \frac{x}{3} \leq t < \infty. \end{cases}$$

if $x > 4$, then

$$T(x)(t) = \begin{cases} v, & \text{if } t = \frac{1}{6} \\ a, & \text{if } t \neq \frac{1}{6} \end{cases}$$

Suppose that $\alpha_L(x) = v$ for each $x \in X$, then

$$[Tx]_{\alpha_L(x)} = \begin{cases} \{0\}, & \text{if } x = 0 \\ [0, \frac{x}{17}], & \text{if } x \in (0, 4] \\ \{\frac{1}{6}\}, & \text{if } x > 4. \end{cases}$$

Obviously, $[Tx]_{\alpha_L(x)} \in CB(X)$ for each $x \in X$.

Define the mappings $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and

$\eta : X^2 \rightarrow \mathbb{R}^+$ respectively as follows:

$$\psi(t) = \begin{cases} \frac{t}{5}, & \text{if } t \in [0, 4] \\ \frac{1}{5}, & \text{if } t > 4. \end{cases}$$

$$\eta(x, y) = \begin{cases} 3, & \text{if } x, y \in [0, 4] \\ \max\{x, y\}, & \text{if } x, y \in (4, \infty). \end{cases}$$

Now, we verify Conditions $(C_2), (C_3)$ and (C_4) of Theorem 2.9. Let $x, y \in (0, 4]$ and $\eta(x, y) \geq 1$. Then, $\eta^*([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = \inf\{\eta(p, q) : p \in [Tx]_{\alpha_L(x)}, q \in [Ty]_{\alpha_L(y)}\} = 3$;

that is, T is η^* -admissible. Hence, (C_2) holds good. Take $x_0 \in (0, 4]$, then $[Tx_0]_{\alpha_L(x_0)} = [0, \frac{x_0}{17}]$. If $x_1 \in [Tx_0]_{\alpha_L(x_0)}$, then $x_1 \in [0, \frac{x_0}{17}] \subseteq [0, 4]$ and $\eta(x_0, x_1) = 3$. Thus, (C_3) is satisfied. Now, consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $\eta(x_n, x_{n+1}) \geq 1$ such that $\lim_{n \rightarrow \infty} x_n = x \in X$. Then, we are sure that $x_n \in [0, 4]$. It follows that $x \in [0, 4]$. Therefore, $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$, proving (C_4) . To see that T is an η^* - (ψ, K) weak L -fuzzy contraction, first observe that for $x = y = 0$ or $x, y \in (4, \infty)$, there is nothing to show, since

$$\begin{aligned} H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) &= H(\{0\}, \{0\}) = 0 \\ &= H\left(\left\{\frac{1}{6}\right\}, \left\{\frac{1}{6}\right\}\right). \end{aligned}$$

So, if $\eta(x,y) \geq 1$ and $x,y \in [0,4]$ with $x \neq y$, we consider the following cases: Case

I. If $x < y$, then $H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = H([0, \frac{x}{17}], [0, \frac{y}{17}]) = \frac{y}{17}$, $d(x,y) = y$, $d(x, [Tx]_{\alpha_L(x)}) = x$, $d(y, [Ty]_{\alpha_L(y)}) = y$, $d(y, [Tx]_{\alpha_L(x)}) = y$, and

$$d(x, [Ty]_{\alpha_L(y)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{17} \\ x, & \text{if } x > \frac{y}{17}. \end{cases}$$

So, for each $K \geq \frac{1}{5}$ it is easy to check that the contraction Condition (2.2) is satisfied.

Case II. If $x,y \in [0,4]$ and $x > y$, then $H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = H([0, \frac{x}{17}], [0, \frac{y}{17}]) = \frac{x}{17}$, $d(x,y) = x$, $d(x, [Tx]_{\alpha_L(x)}) = x$, $d(y, [Ty]_{\alpha_L(y)}) = y$, and

$$d(y, [Tx]_{\alpha_L(x)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{17} \\ y, & \text{if } x > \frac{y}{17}. \end{cases}$$

Thus, the inequality (2.2) holds for all $K \geq \frac{1}{5}$. It follows that T is an $\eta^*(\psi, K)$ -weak L -fuzzy contraction. Therefore, all the hypotheses of Theorem 2.9 are satisfied. Hence, there exists $u = 0 \in X$ and $\alpha_L(0) = v$ such that $0 \in [T0]_{\alpha_L(0)}$.

Consequences in ordered metric spaces

The study of existence of fixed points on metric spaces endowed with a partial order is one of the very interesting improvements in the area of fixed point theory. This trend was introduced by (Turinici, 1986) in 1986, but it became one of the core research subjects after the publications of the results of (Ran & Reurings, 2004) and (Nieto & Rodriguez, 2005).

In this section, we study the analogue of our main result in the setting of ordered metric spaces. Some preliminary concepts, introduced as follows, are

necessary. Accordingly, (X, d, \preceq) is called an ordered metric space, if:

- (i) (X,d) is a metric space, and
- (ii) (X, \preceq) is a partially ordered set (poset).

Definition 3.1. Let (X, \preceq) be a poset, $A, B \subseteq X$ and $T : X \rightarrow L^X$ be an L -fuzzy set-valued map. Then, we write $A \vee B$ if an only if $a \preceq b$ for all $a \in A$ and $b \in B$. The mapping T is called order preserving if for each $x,y \in X$, there exist $\alpha_L(x), \alpha_L(y) \in L \setminus \{0_L\}$ such that $x \preceq y$ implies $[Tx]_{\alpha_L(x)} \vee [Ty]_{\alpha_L(y)}$.

Definition 3.2. Let (X, d, \preceq) be an ordered metric space. An L -fuzzy set-valued map $T : X \rightarrow L^X$ is called an ordered (ψ, K) -weak L -fuzzy contraction, if for all $x,y \in X$, there exist $\alpha_L(x), \alpha_L(y) \in L \setminus \{0_L\}$ with $[Tx]_{\alpha_L(x)} \vee [Ty]_{\alpha_L(y)}$, $\psi \in \Xi$ and $K \geq 0$ such that

$$H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \psi(\Omega_L T(x,y)) + Kd(x, [Tx]_{\alpha_L(x)}). \tag{3.1}$$

Consistent with (Shukla, 2017), we say that a sequence $\{x_n\}_{n \in \mathbb{N}}$ is called an ordered sequence if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. The class of all ordered sequences in X is denoted by $E_{X, \preceq}$. For an L -fuzzy set-valued map $T : X \rightarrow L^X$, we denote by $G^{\preceq}(T)$, the \preceq -graph of T , where

$$G^{\preceq}(T) = \{(x_n, x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_{n \in \mathbb{N}} \in E_{X, \preceq} \cap T_{r(x_1)}\}.$$

The L -fuzzy set-valued map T is said to be \preceq -closed, if $G^{\preceq}(T)$ is a closed subset of X^2 . The metric space (X, d, \preceq) is called \preceq -complete, if every Cauchy sequence in $E_{X, \preceq}$ is convergent in X^2 .

The following result is an extension of the main results of Ran and Reurings (2004) and Nieto and Rodriguez (2005) into L -fuzzy domain.

Corollary 3.3. Let (X, d, \preceq) be an \preceq -complete metric space and $T : X \rightarrow L^X$ be an ordered (ψ, K) -weak L -fuzzy

contraction. Assume that the following conditions are satisfied:

(C₁) for each $x \in X$, there exists $\alpha_L(x) \in L \setminus \{0_L\}$ such that $[Tx]_{\alpha_L(x)} \in CB(X)$;

(C₂) T is order preserving;

(C₃) there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ with $x_0 \preceq x_1$;

(C₄) at least one of the following hypotheses holds:

(i) for a trajectory $\{x_n\}_{n \in \mathbb{N}} \in E_{X \rightarrow}$ starting at x_1 and converging to $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;

(ii) T is an η -closed L -fuzzy set-valued map.

Then, there exists $u \in X$ and $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$. *Proof.* Consider a mapping $\eta : X^2 \rightarrow \mathbb{R}^+$ defined by

$$\eta(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$$

If $\eta(x, y) \geq 1$, then, by order preserveness of T , $[Tx]_{\alpha_L(x)} \vee [Ty]_{\alpha_L(y)}$ for some $\alpha_L(x), \alpha_L(y) \in (0, 1]$. Hence,

$$\eta^*([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = \inf\{\eta(p, q) : p \in [Tx]_{\alpha_L(x)}, q \in [Ty]_{\alpha_L(y)}\} = 1.$$

This implies that T is an η^* -admissible L -fuzzy set-valued map. Moreover, from Condition (C₃), there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ such that $\eta(x_0, x_1) \geq 1$. And, by (i) of Condition (C₄), for any trajectory $\{x_n\}_{n \in \mathbb{N}} \in E_{X \preceq}$ converging to $x \in X$, we must have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. In addition, it is clear that η -closedness of T implies the η -closedness of T and η -completeness of (X, d, \preceq) yields η -completeness. Since T is an ordered (ψ, K) -weak L -fuzzy contraction, by definition of η , the mapping T is an η^* - (ψ, K) -weak L -fuzzy contraction. Consequently,

Theorem 2.9 or 2.12 can be availed to obtain $u \in X$ and some $\alpha_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha_L(u)}$.

CONCLUSION

Based on the ideas of L -fuzzy sets, a new notion of L -fuzzy set-valued maps under the name η^* - (ψ, K) -weak L -fuzzy contractions is initiated and sufficient criteria for existence of L -fuzzy fixed points for such contractions in the framework of complete metric spaces are investigated in this note. As some consequences of the obtained results herein, a few related fixed point theorems in ordered metric spaces are underlined and analyzed. An example is constructed to support the hypotheses of our established results. It is observed conclusively that the concepts proposed in this work harmonize, improve and complement some recently published results regarding η - ψ -contractive mappings and their corresponding fixed point theorems.

Competing Interests

The author declares that there is no competing interests.

Acknowledgement

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that helped to improve this manuscript.

REFERENCES

- Ali, M. U. & Kamran, T. (2013). On $(\alpha_* - \psi)$ -contractive multi-valued mappings. *Fixed Point Theory and Applications*, 2013, Article ID 137.
- Asl, J. H., Rezapour, S. & Shahzad, N. (2012). On fixed points of $\alpha - \psi$ -contractive multifunctions. *Fixed Point Theory and Applications*, 2012(1), 1-6.
- Azam, A., Arshad, M. & Vetro, P. (2010). On a pair of fuzzy φ -contractive mappings. *Mathematical and Computer Modelling*, 52(2), 207-214.
- Azam, A. & Beg, I. (2009). Common fixed points of fuzzy maps. *Mathematical and computer modelling*, 49(7), 1331-1336.
- Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1), 133-181.
- Berinde, V. (2004). Approximating fixed points of weak contractions using the Picard iteration. In *Nonlinear Analysis Forum*, 9(3), 43-54.
- El Naschie, M. S. (2002). Wild topology, hyperbolic geometry and fusion algebra of high energy particle physics. *Chaos, Solitons and Fractals*, 13(9), 1935-1945.
- El Naschie, M. S. (2000). On the unification of the fundamental forces and complex time in the E^∞ space. *Chaos, Solitons and Fractals*, 11(7), 1149-1162.
- Heilpern, S. (1981). Fuzzy mappings and fixed point theorem. *Journal of Mathematical Analysis and Applications*, 83(2), 566-569.
- Hussain, N., Ahmad, J. & Azam, A. (2014). Generalized fixed point theorems for multi-valued $\alpha - \psi$ -contractive mappings. *Journal of Inequalities and Applications*, 2014(1), 1-15.
- Hussain, N, Salimi, P. & Latif, A. (2013 a). Fixed point results for single and set-valued $\alpha - \psi$ -contractive mappings. *Fixed Point Theory and Application*, Article ID 212.
- Hussain, N, Karapınar, E, Salimi, P. & Akbar, F. (2013 b). α -Admissible mappings and related fixed point theorems. *Journal of Inequalities and Applications*, 2013, Article ID 114.
- Goguen, J. A. (1967). L-fuzzy sets. *Journal of mathematical analysis and applications*, 18(1), 145-174.
- Kutbi, M.A., Ahmad, J & Azam, A. (2013). On fixed points of $\alpha - \psi$ -contractive multi-valued mappings in cone metric spaces. *Abstract and Applied Analysis*, 2013, Article ID 313782.
- Mohammed, S. S. & Azam, A. (2019). Fixed points of soft-set valued and fuzzy set-valued maps with applications. *Journal of Intelligent and Fuzzy Systems*, 37(3), 3865-3877.

- Mohammed, S. S., Fulatan, I. A. & Sirajo, Y. (2021). Fixed Points of p -Hybrid L-Fuzzy Contractions. *Sahand Communications in Mathematical Analysis*, 18(3), 1-25.
- Mohammed, S. S. (2021). Fixed Point Results for Fuzzy Set-Valued Maps on Metric Space. *Unilag Journal of Mathematics*, 1(1), 20-34.
- Mohammed, S. S., Saima, R., Khadija, M. A. & Monairah, A. (2020). On nonlinear fuzzy set-valued Θ -contractions with applications. *AIMS Mathematics*, 6(1), 10431-10448.
- Nadler, S.B. (1969). Multi-valued contraction mappings. *Pacific Journal of Mathematics*, 30, 475-478.
- Nieto, J. J. & Rodríguez-López, R. (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order*, 22(3), 223-239.
- Park, J. Y. & Jeong, J. U. (1997). Fixed point theorems for fuzzy mappings. *Fuzzy Sets and Systems*, 87(1), 111-116.
- Ran, A. C. and Reurings, M. C. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proceedings of the American Mathematical Society*, 5(2), 1435-1443.
- Rasham, T., Asif, A., Aydi, H. & De La Sen, M. (2021a). On pairs of fuzzy dominated mappings and applications. *Advances in Difference Equations*, 2021(1), 1-22.
- Rasham, T., Marino, G., Shahzad, A., Park, C. & Shoaib, A. (2021b). Fixed point results for a pair of fuzzy mappings and related applications in b -metric like spaces. *Advances in Difference Equations*, 2021(1), 1-18.
- Rasham, T., Shoaib, A., Park, C., Agarwal, R. P. & Aydi, H. (2021c). On a pair of fuzzy mappings in modular-like metric spaces with applications. *Advances in Difference Equations*, 2021(1), 1-17.
- Rasham, T., Shabbir, M. S., Agarwal, P. & Momani, S. (2021d). On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications. *Fuzzy Sets and Systems*, <https://doi.org/10.1016/j.fss.2021.09.002>.
- Rashid, M., Azam, A., & Mehmood, N. (2014a). L-Fuzzy fixed points theorems for L-fuzzy mappings via β_{FL} -admissible pair. *The Scientific World Journal*, Vol. 2014, Article ID 853032, 8 pages.
- Rashid, M., Kutbi, M. A., & Azam, A. (2014b). Coincidence theorems via alpha cuts of L-fuzzy sets with applications. *Fixed Point Theory and Applications*, 2014(1), 212.

- Samet, B., Vetro, C. & Vetro, P. (2012). Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Analysis: Theory, Methods and Applications*, 75(4), 2154-2165.
- Shagari, M. S. & Azam, A. (2020). Fixed point theorems of fuzzy set-valued maps with applications, *Problemy Analiza*, 9(27), 68–86.
- Shukla, S., Gopal, D. & Mart'inez-Moreno, J. (2017). Fixed points of set-valued F -contractions and its application to non-linear integral equations. *Filomat*, 31(11), 3377-3390.
- Turinici, M. (1986). Abstract comparison principles and multivariable Gronwall-Bellman inequalities. *Journal of Mathematical Analysis and Applications*, 117(1), 100-127.
- Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3), 3