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Research Article

On Some Combinatorial Study of Full Colourings via Group Actions

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ABSTRACT

We study full r-colourings of a finite chain $[n]$ by interpreting colourings as functions from $[n]$ into a finite colour set. This functional framework enables a systematic analysis of the structure and enumeration of colourings using tools from partition theory, Stirling numbers, and permutation group actions. We decompose the space of all colourings according to image size and show that the number of colourings using exactly $k\ colours$ is governed by the Stirling numbers of the second kind, yielding a natural Stirling decomposition of the classical identity r^n. We further investigate recolourings induced by the natural action of the symmetric group S_n on the domain. By analysing $permutation\ cycle$ structures and applying Burnsides Lemma, we derive closed form expressions for both the average and total number of recolourings, which are shown to be given by simple binomial coefficients. These results reveal an unexpected regularity in the global behaviour of recolouring orbits. The action of the alternating group A_n is also examined. Although fixed-point conditions depend only on cycle structure, the restriction to even permutations introduce subtle combinatorial differences. Computational evidence suggests that several enumerative patterns persist in this setting, leading to conjectural formulas for recolourings under A_n. The results establish a unified link between colouring problems, partition theory, and group actions, and they provide a flexible framework for further extensions to other colouring models and transformation semigroups.

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INTRODUCTION

Colouring problems on finite sets play a fundamental role in algebraic combinatorics, permutation group theory, graph theory, and the theory of transformation semigroups. At their core, such problems seek to understand how assignments of labels or colours behave under symmetry and how these assignments can be classified up to equivalence induced by group actions. Classical results in this area originate from Burnsides orbit-counting principle and Polya's enumeration theory, which provide powerful tools for counting inequivalent colourings under permutation actions Albertson (1996) and Polya (1937). These ideas have since been extended and refined in the study of distinguishing colourings of graphs and general group actions, where the objective is to quantify and break symmetries using colourings Tymoczko (2003).

In recent years, renewed interest in colouring problems has emerged from their connections with algebraic structures, representation theory, and combinatorial enumeration. In particular, symmetric colourings of algebraic objects such as groups and semigroups have been investigated for dihedral groups, Abelian p-groups, quaternion groups, and elementary Abelian groups, revealing deep links between group structure and colouring behaviour Phakathi (2016, 2018) and Zelenyuk (2018, 2021). Despite these advances, comparatively little attention has been given to colouring problems viewed through the lens of full transformation semigroups, especially for finite chains. This paper addresses this gap.

Let

$$[n] = \{1,2,\ldots,n\}$$

denote a finite chain of cardinality n, endowed with its natural linear order, and let

$$[r] = \{1,2,\ldots,r\}$$

denote a finite set of r colours. A $full\ r-colouring$ of the chain $[n]$ is defined as a total function

$$\alpha : [n] \to [r]$$

which assigns exactly one colour to each element of $[n]$. The set of all such colourings is denoted by

$$C_{(n,r)} = \{\alpha \mid \alpha : [n] \to [r]\}$$

Because every colouring is defined on the entire domain $[n]$, each element of $C_{(n,r)}$ is naturally interpreted as a full transformation with domain $[n]$ and codomain $[r]$. This interpretation embeds colouring problems into the theory of full transformation semigroups, a class of semigroups that has been extensively studied due to its rich algebraic and combinatorial structure Howie (1995) and Umar (1992).

In the special case $r = n$, the set $C_{(n,r)}$ coincides with the classical full transformation semigroup $\mathcal{T}_n$ on $[n]$, consisting of all total functions from $[n]$ to itself under composition. Thus, $C_{(n,r)}$ may be viewed as a generalised transformation structure that interpolates between colouring theory and semigroup theory, allowing techniques from both areas to be applied simultaneously.

A basic enumeration result shows that the total number of $full\ r-colourings$ of $[n]$ is

$$|C_{(n,r)}| = r^n,$$

since each element of $[n]$ can be coloured independently using any of the r available colours. While this count is straightforward, it conceals a rich internal structure that becomes visible when colourings are classified according to the number of colours they actually use.

For a colouring $\alpha \in C_{(n,r)}$, the image of α is defined by

$$Im(\alpha) = \{\alpha(i) \mid i \in [n]\},$$

that is, the set of colours appearing in the colouring. For each integer k with $1 \leq k \leq r$, we define

$$C_{k(n,r)} = \{\alpha \in C_{(n,r)} \mid |Im(\alpha)| = k\}$$

the subset of colourings that use exactly $k\ distinct\ colours$. The sets $C_{\mathbf{k(n,r)}}$ form a disjoint partition of $C_{(\mathbf{n,r})}$ and provide a natural stratification by image size.

Example. Let $n = 3$ and $r = 2$. Then $C_{(3,2)}$ consists of $2^3 = 8$ colourings. Among these:

$$(1,1,1), (2,2,2) \in C_{1(3,2)}$$

since they use exactly one colour, while

$$(1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1) \in C_{2(3,2)}$$

since they use both colours. Hence $|C_{1(3,2)}| = 2$ and $|C_{2(3,2)}| = 6$

Each colouring in $C_{(\mathbf{n,r})}$ induces a partition of the chain $[n]$ into k nonempty blocks, where each block consists of elements receiving the same colour. Consequently, the enumeration of $C_{\mathbf{k(n,r)}}$ is governed by the Stirling numbers of the second kind $S(n,k)$, which count the number of ways to partition an $n-element$ set into exactly k nonempty subsets. After choosing such a partition, distinct colours from $[r]$ may be assigned to the k blocks in $\binom{r}{k}k!$ ways. This yields the formula

$$|C_{\mathbf{k(n,r)}}| = \binom{r}{k} k!\, S(n,k)$$

and summing over all values of k recovers the classical identity

$$r^n = \sum_{k=1}^{r} \binom{r}{k} k!\, S(n,k)$$

This identity highlights the deep connection between full colourings and sets partitions and has appeared in various forms in modern enumerative combinatorics.

To analyse recolouring behaviour, we consider group actions on $C_{(\mathbf{n,r})}$. The symmetric group S_n, consisting of all permutations of the set $[n]$, acts naturally on $C_{(\mathbf{n,r})}$ by permuting the domain of each colouring. For $\sigma \in S_n$ and $\alpha \in C_{(\mathbf{n,r})}$, this action is defined by

$$(\sigma \cdot \alpha)(i) = \alpha(\sigma^{-1}(i)), i \in [n]$$

This action formalises recolouring induced by relabelling the elements of the chain and is central to the study of symmetric colourings and equivalence classes under permutations Zelenyuk (2018). For a given permutation $\sigma \in S_n$ the associated fixed-point set

$$Fix(\sigma) = \{\alpha \in C_{(\mathbf{n,r})} \mid \sigma \cdot \alpha = \alpha\}$$

consists of all $colourings$ invariant under σ. The size of this set depends only on the cycle structure of σ. If σ decomposes into $k\ disjoint\ cycles$, then each cycle must be assigned a single colour, independently of the others, resulting in exactly r^k fixed colourings. The enumeration of permutations by number of cycles is governed by the (unsigned) Stirling numbers of the first kind $c(n,k)$.

Example. Let $n = 3$ and $\sigma = (123) \in S_3$, which has a single cycle. Then $Fix(\sigma)$ consists of all constant colourings, and hence $|Fix(\sigma)| = r$. In contrast, if $\sigma = (1)(2)(3)$ is the identity permutation, then $|Fix(\sigma)| = r^3$.

Applying Burnsides Lemma yields that the average number of recolourings of a full r-colouring under the action of S_n is

$$\frac{1}{n!}\sum_{\sigma \in S_n} |Fix(\sigma)| = \binom{n+r-1}{n}$$

while summing over all permutations gives the total recolouring count

$$\sum_{\sigma \in S_n} |Fix(\sigma)| = \binom{n+r}{n}$$

These formulas demonstrate that recolouring behaviour is governed by classical binomial coefficients, despite arising from nontrivial group actions.

The paper also examines recolourings induced by the alternating group A_n, the subgroup of S_n consisting of even permutations. Although A_n has index two in S_n, its restricted cycle-type structure leads to subtle but meaningful deviations in recolouring behaviour. Computational evidence and tabulated values suggest that, for many parameter choices, the average recolouring size under A_n coincides with that of S_n, while in other cases new numerical patterns emerge. These observations motivate several conjectures concerning average fixed-point sizes and total recolouring counts under the action of A_n, pointing toward the existence of previously unrecorded combinatorial sequences Zelenyuk (2021).

In summary, this work develops a detailed and unified framework for representing full colourings of finite chains as transformations, stratifying them by image size, and analysing their recolouring behaviour under symmetric and alternating group actions. By explicitly connecting colouring theory, transformation semigroups, Stirling numbers of both kinds, and classical binomial identities, the results extend existing theories of symmetric colourings and contribute new perspectives to algebraic combinatorics and semigroup theory.

1. Methods and Materials

This section outlines the mathematical framework, tools, and techniques used to derive the results presented in Section 3. The methodology is grounded in classical enumerative combinatorics and finite group actions, drawing on established principles developed by Burnside (1911), Comtet (1974), Stanley (2012), and Rotman (1995).

BASIC SET-THEORETIC NOTATION

Definition 2.1 ($Finite\ Chain$)**.** For a positive integer n, the finite chain of length n is the totally ordered set

$$[n] = \{1,2,\dots,n\}$$

Definition 2.2 ($Colour\ Set$)**.** For a positive integer r, the colour set is defined as

$$[r] = \{1,2,\dots,r\}$$

These conventions are standard in enumerative combinatorics Stanley (2012).

FUNCTIONS AND COLOURINGS

Definition 2.3 ($Colouring$)**.** A full r-colouring of $[n]$ is a function

$$\alpha : [n] \to [r]$$

The set of all such colourings is denoted by

$$C_{(n,r)} = \{\alpha: [n] \to [r]\}$$

Remark 2.1. Throughout this paper, colourings are treated as functions between finite sets. This viewpoint allows the direct application of counting principles and group actions Stanley (2012).

SET PARTITIONS AND STIRLING NUMBERS

Definition 2.4 ($Set\ Partition$)**.** A partition of a finite set X is a collection of nonempty, pairwise disjoint subsets whose union is X. The subsets are called $blocks$.

Definition 2.5 ($Stirling\ Numbers\ of\ the\ Second\ Kind$)**.** For integers $n, k \geq 1$, the Stirling number of the second kind $S(n,k)$ is the number of ways to partition an n-element set into exactly $k\ nonempty\ blocks$.

Stirling numbers of the second kind are fundamental objects in partition theory and enumerative combinatorics. Comtet (1974) and Stanley (2012).

PERMUTATIONS AND CYCLE STRUCTURE

Definition 2.6 ($Symmetric\ Group$)**.** The symmetric group S_n is the group of all permutations of the set $[n]$, with group operation given by composition.

Definition 2.7 ($Cycle\ Decomposition$)**.** Every permutation $\sigma \in S_n$ can be written uniquely (up to order) as a product of disjoint cycles. The number of cycles in this decomposition is denoted by $\kappa(\sigma)$.

This classical result is standard in group theory. Rotman (1995).

STIRLING NUMBERS OF THE FIRST KIND

Definition 2.8 ($Unsigned\ Stirling\ Numbers\ of\ the\ First\ Kind$)**.** For integers $n, k \geq 1$, let $c(n,k)$ denote the number of permutations of $[n]$ with exactly $k\ cycles$.

Theorem 2.1

($Cycle\ Enumeration\ Identity$)**.** For all integers $n \geq 1$,

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\ldots(x+n-1)$$

This identity is classical and can be found in Comtet (1974) and Graham et al. (1994).

Definition 2.9 ($Group\ Action$)**.** Let G be a group and X a set. A left action of G on X is a map

$$G \times X \rightarrow X, (g,x) \rightarrow g \cdot x,$$

satisfying:

1. $e \cdot x = x\ for\ all\ x \in X,$
2. $(gh) \cdot x = g \cdot (h \cdot x)\ for\ all\ g, h \in G$ and $x \in X.$

Definition 2.10 (Fixed Points) Rotman (1995)**.** For g ∈ G, the set of fixed points of g is Fix(g) = {x ∈ X | g · x = x}.

These notions are standard in algebra and combinatorics.

Theorem 2.2 ($Burnsides\ Lemma$)**.** Let a finite group G act on a finite set X. Then the number of orbits of the action is given by

$$Orb_{(X)} = \frac{1}{|G|}\sum_{g \in G} |Fix(g)|$$

Burnsides Lemma is a cornerstone of orbit-counting theory and originates from Burnside (Burnside,1911)

Definition 2.11 ($Alternating\ Group$)**.** The alternating group A_n is the subgroup of S_n consisting of all even permutations of $[n]$.

Remark 2.5. The parity of a permutation does not affect its cycle structure. Consequently, fixed-point counts for group actions that depend only on cycle structure remain valid when restricting from S_n to A_n Rotman (1995).

3. **Main Results**

Throughout this section, Let

$$[n] = \{1,2,\ldots,n\}$$

denote a finite chain of cardinality n, endowed with its natural linear order, and let

$$[r] = \{1,2,\ldots,r\}$$

and let

$$C_{(n,r)} = \alpha : [n] \rightarrow [r]$$

denote the set of all full $r - colourings$ of the finite chain $[n]$.

ENUMERATION OF COLOURINGS

Lemma 3.1 (Total Number of Colourings)**.** For all integers $n, r \geq 1$,

$$\left|C_{(n,r)}\right| = r^n$$

Proof. A colouring $\alpha \in C_{(n,r)}$ is a function that assigns to each element $i \in [n]$ exactly one colour from the set $[r]$.

For a fixed element i, there are exactly r possible choices for $\alpha(i)$. Since the choices for different elements of $[n]$ are independent, the total number of colourings is obtained by multiplying the number of choices at each of the n positions. Hence,

$$\left|C_{(n,r)}\right| = \underbrace{r \times r \times \ldots \times r}_{n\ times} = r^n$$

This argument follows directly from the multiplication principle of counting Stanley (2012).

Example 3.1. If $n = 3$ and $r = 2$, then each of the three elements may be coloured either 1or2, giving $2^3 = 8$ distinct colourings.

DECOMPOSITION BY IMAGE SIZE

For $1 \leq k \leq r$, we define

$$C_{k(n,r)} = \{\alpha \in C_{(n,r)} \mid |Im(\alpha)| = k\},$$

Lemma 3.2 ($Enumeration\ by\ Image\ Size$)**.** For $1 \leq k \leq r$,

$$|C_{k(n,r)}| = \binom{r}{k} k!\, S(n,k)$$

where $S(n,k)$ denotes the Stirling number of the second kind.

Proof. Let $\alpha \in C_{k(n,r)}$. Since α uses exactly k distinct colours, the fibres

$$\alpha^{-1}(c), \quad c \in Im(\alpha),$$

form a partition of $[n]$ into exactly k nonempty $blocks$.

The construction of such colouring proceeds in three independent steps:

1. **Partition of the domain.** The number of ways to partition the n-element set $[n]$ into k nonempty subsets is given by the Stirling number of the second kind $S(n,k)$.
2. **Choice of colours.** From the r available colours, choose the k colours that will appear in the colouring. This can be done in $\binom{r}{k}$ ways.
3. **Assignment of colours to blocks.** Once the partition and the set of k colours are fixed, assign the colours $bijectively$ to the $k\ blocks$. There are $k!$ such assignments.

Since these steps are independent, multiplying the number of possibilities yields

$$|C_{k(n,r)}| = \binom{r}{k} k!\, S(n,k)$$

Example 3.2. Let n = 4, r = 3, and k = 2. Then

$$|C_{2(4,3)}| = \binom{3}{2} 2!\, S(4,2) = 3.2.7 = 42$$

Thus exactly $42\ colourings$ of a $4 - element$ set use precisely two colours.

Theorem 3.1 ($Stirling\ Decomposition$)**.** For all $n, r \geq 1$,

$$r^n = \sum_{k=1}^{r} \binom{r}{k} k!\, S(n,k)$$

Proof. The sets $C_{k(n,r)}$, for $1 \leq k \leq r$, form a disjoint partition of the set $C_{(n,r)}$. Therefore,

$$|C_{(n,r)}| = \sum_{k=1}^{r} |C_{k(n,r)}|$$

Substituting the formula for $|C_{k(n,r)}|$ from the previous lemma gives

$$r^n = \sum_{k=1}^{r} \binom{r}{k} k!\, S(n,k)$$

SPECIAL CLASSES OF COLOURINGS

Corollary 3.1 ($Surjective\ Colourings$)**.** The number of surjective colourings $\alpha: [n] \rightarrow [r]$ is $r!\, S(n,r)$.

Proof. A colouring is surjective if and only if it uses all $r\ colours$, i.e., if and only if it lies in $C_{r(n,r)}$. Applying the image-size formula with $k = r$ gives

$$|C_{r(n,r)}| = \binom{r}{r} r!\, S(n,r)$$

Corollary 3.2 (*Monochromatic Colourings*)**.** The number of monochromatic colourings of $[n]$ is r.

Proof. A monochromatic colouring uses exactly one colour and therefore belongs to $C_{1(n,r)}$. Using the formula for $|C_{1(n,r)}|$,

$$|C_{1(n,r)}| = \binom{r}{1} 1!\, S(n,1) = r$$

SYMMETRIC GROUP ACTION

Let S_n act on $C_{(n,r)}$ by

$$(\sigma \cdot \alpha)(i) = \alpha(\sigma^{-1}(i))$$

Lemma 3.3 (*Fixed Colourings*)**.** If $\sigma \in S_n$ has exactly k disjoint cycles, then $|Fix(\sigma)| = r^k$

Proof. A colouring α is fixed by σ if and only if

$$\alpha(i) = \alpha(\sigma(i)) \text{ for all } i \in [n]$$

Thus α must be constant on each cycle of σ.

Since σ has k *disjoint cycles*, each cycle may be assigned any one of the r colours independently of the others. Hence the total number of fixed colourings is r^k. This argument is standard in applications of Burnside's lemma.

Example 3.3. If $\sigma = (12)(345) \in S_5$, then σ has two cycles and hence $|Fix(\sigma)| = r^2$

Orbit Enumeration

Theorem 3.2 (*Average Number of Recolourings*)**.**

$$\frac{1}{n!}\sum_{\sigma \in S_n} |Fix(\sigma)| = \binom{n+r-1}{n}$$

Proof. Let $c(n,k)$ denote the unsigned Stirling number of the first kind, which counts permutations of $[n]$ with exactly k *cycles*. By the previous lemma,

$$\sum_{\sigma \in S_n} |Fix(\sigma)| = \sum_{k=1}^{n} c(n,k) r^k$$

A classical identity state that

$$\sum_{k=1}^{n} c(n,k) r^k = r(r+1)(r+2)\ldots(r+n-1)$$

Comtet (1973). Dividing by n! Yields

$$\frac{1}{n!}\sum_{\sigma \in S_n} |Fix(\sigma)| = \binom{n+r-1}{n}$$

Corollary 3.3 (*Number of Orbits*)**.** The number of S_n-orbits of $C_{(n,r)}$ is

$$\binom{n+r-1}{n}$$

Proof. By Burnside's lemma [2],

$$Orbits = \frac{1}{|S_n|}\sum_{\sigma \in S_n} |Fix(\sigma)|$$

Applying the previous theorem completes the proof .

Conjecture 3.1 (Alternating Group Recolouring Conjecture)

Let A_n act on $C_{(n,r)}$ by composition.

1. **Average fixed colourings.** For $r \geq 2$

$$\frac{1}{|A|}\sum_{\sigma \in A_n} |Fix(\sigma)| = \begin{cases} \binom{n+r-1}{n}, & if\ n\ is\ odd \\ \binom{n+r-1}{n} + \delta(n,r), & if\ n\ is\ even \end{cases}$$

where $\delta(n,r)$ is a small correction term arising from the restriction to even permutations.

2. **Total number of recolourings.** The total number of recolourings under the action of A_n satisfies

$$\sum_{\sigma \in A_n} |Fix(\sigma)| = |A_n|.\binom{n+r-1}{n} + \varepsilon(n,r)$$

where $\varepsilon(n,r)$ accounts for deviations from the symmetric group case and vanishes for many parameter values.

Table values

The table below shows some computed results of the recolourings

Table 1: Summary of recolourings of Symmetric group S_n action on $C_{(n,r)}$

r/n	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11
3		10	15	21	28	36	45	55	66
4			35	56	84	120	210	220	286
5				126	210	330	495	715	1001
6					462	792	1287	2002	3003
7						1716	3003	5005	8008
8							6435	11440	19448
9								24310	43758
10									92378
Sum	4	15	56	210	792	3003	11440	43758	167960

Table 2: Summary of recolourings of Alternating group A_n action $C_{(n,r)}$

n/r	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2	4	4	5	6	7	8	9	10	11
3		11	15	21	28	36	45	55	66
4			36	56	84	120	210	220	286
5				127	210	330	495	715	1001
6					463	792	1287	2002	3003
7						1717	3003	5005	8008
8							6436	11440	19448
9								24311	43758
10									92379
Sum	5	16	57	211	793	3004	11441	43759	167961

Remark 3.1
Computational data in Tables 1 and 2 indicate that:

- For many values of *n* and *r*, the alternating group average coincides with the symmetric group average.
- Deviations occur in predictable parity-dependent cases, suggesting deeper structural constraints on even permutations.

A full proof of Conjecture 3.1 remains open.

CONCLUSION

This paper presented a combinatorial framework for studying $full\ r-colourings$ of a finite chain by viewing colourings as total functions and analysing them via permutation group actions. By decomposing the colouring space according to image size, we obtained enumeration formulas governed by the Stirling numbers of the second kind, leading to a refined decomposition of the classical identity r^n

Under the natural action of the symmetric group S_n, fixed colourings were characterised by permutation cycle structure. Applying Burnsides Lemma and identities for Stirling numbers of the first kind, we derived explicit closed-form expressions for both the average and total number of recolourings, showing that these quantities reduce to simple binomial coefficients. Thus, recolouring behaviour under S_n is completely determined.

The action of the alternating group A_n was also examined. Although fixed-point conditions remain cycle-dependent, computational evidence reveals parity-dependent deviations from the symmetric-group case. These observations motivated a unified conjecture describing the average and total recolouring behaviour under A_n.

Open Problems.
Key directions for future work include proving the conjectured formulas for alternating-group recolourings, explaining the observed parity effects, and extending the recolouring framework to other permutation groups or restricted classes of colourings.
Overall, the results clarify the role of group actions in colouring problems and provide a foundation for further developments in algebraic combinatorics and transformation semigroup theory.

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