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Research Article

Weighted Lacunary I_2 −Statistical Convergence for Double Sequences

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ABSTRACT
Statistical convergence and its ideal based generalizations have become an active area of summability theory since their independent introduction by Fast and Schoenberg, with subsequent extensions to double sequences, weighted means, and lacunary sequences by several authors. Building on the concept of weighted lacunary I-statistical convergence recently introduced for single sequences, this paper extends the notion to double sequences by defining weighted lacunary I_2-statistical convergence and the associated weighted lacunary I_2-summability method $[R^2, \beta_m, \gamma_n, \theta_{r,s}]^{I2}$ for double sequences with respect to an admissible ideal I_2 of $\mathbb{N} \times \mathbb{N}$. Several inclusion relations connecting these two convergence notions are established. It is shown that weighted lacunary I_2-summability implies weighted lacunary I_2-statistical convergence in general, with the converse holding under a boundedness condition on the weighted double sequence. Relations between the weighted and unweighted forms of lacunary I_2-statistical convergence are also obtained, depending on whether the weight sequences (β_j) and (γ_k) are bounded above or below by unity. Finally, conditions on the double lacunary ratio $q_{r,s}$ are used to relate weighted lacunary I_2-statistical convergence to weighted (non-lacunary) I_2-statistical convergence, showing that the two notions coincide when $\liminf_{r,s} q_{r,s} > 1$ and when $\limsup_{r,s} q_{r,s} < \infty$. These results generalize several known inclusion theorems for statistical and lacunary statistical convergence of double sequences to the weighted, ideal-based setting.

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INTRODUCTION

The concept of statistical convergence was formally introduced by Fast (1951) and Schoenberg (1959) independently. Although statistical convergence was introduced over fifty years ago, it has become an active area of research in recent years. It has been applied in various areas such as summability theory (Fridy and Orhan (1993) and Salat (1980)), topological groups (Cakalli (1996) and Cakalli and Das (2009)), topological spaces (Di Maio and Kocinac (2008)), locally convex spaces (Maddox (1988)), measure theory (Cheng et al. (2008)), Connor and Swardson (1993) and Miller (1995), Fuzzy Mathematics (Nuray and Savas (1995) and Savas (2001). Recently, generalizations of statistical convergence have appeared in the study of strong summability and the structure of ideals of bounded functions, (Connor and Swardson (1994)). Mursaleen and Edely (2003) extended the notion of statistical convergence of single sequence to double sequences. Kostyrko et al. (2000/2001) further extended the idea of statistical convergence to convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences. Das and Savas (2014) introduced and studied statistical and lacunary statistical convergence of order. Quite recently Brono et al. (2017) introduced the concept of statistical and lacunary statistical convergence for double sequence of order analogous to Das and Savas (2014). Karakaya and Chishti (2009) introduced and studied the concept of weighted statistical convergence of single sequences. This concept has been extended to double sequences by many researchers (see Brono and Ali (2016), Cinar and Et (2016), e.t.c. Konca (2017) defined the concept of weighted lacunary statistical convergence for single sequence. In this paper we have further extended this concept to double sequences and examined some inclusion relations.

1. Definitions and Preliminaries

The following definitions will be needed in the sequel.

If X is a non-empty set then a family of set $I \subset P(X)$ is called an ideal in X if and only if (i) $\Phi \in I$; (ii) for each $A, B \in I$ we have $A \cup B \in I$; (iii) for each $A \in I$ and $B \subset A$ we have $B \in I$.

Let X is a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if (i) $\Phi \notin F$; (ii) for each $A, B \in F$ we have $A \cap B \in F$; (iii) for each $A \in F$ and $B \supset A$ we have $B \in F$.

An ideal I is called non-trivial if $I \neq \Phi$ and $X \notin I$.

A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\}: x \in X\}$. Throughout the paper we take I_2 as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal I_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{x\} \times \mathbb{N}$ and $\mathbb{N} \times \{x\}$ belong to I_2 for each $x \in \mathbb{N}$.

For further study we shall take $X = \mathbb{N} \times \mathbb{N}$ and I_2 will denote an ideal of subsets of $\mathbb{N} \times \mathbb{N}$. The following proposition expresses a relation between the notions of an ideal and a filter.

Let $I_2 \subset P(\mathbb{N} \times \mathbb{N})$ be a non- trivial ideal. Then the class $F = F(I) = \{M \subset \mathbb{N} \times \mathbb{N}: M = \mathbb{N} \times \mathbb{N} - A, for\ some\ A \in I\}$ is a filter on $\mathbb{N} \times \mathbb{N}$ (we shall call $F = F(I)$ $the\ filter\ associated\ with\ I$).

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}$$

In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as $\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K)$.

For example, let $K = \{(i^2, j^2): i, j \in \mathbb{N}\}$.Then,

$\delta_2 = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$

i.e., the set K has double natural density zero, while the set $K = \{(i, 2j): i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

Note that, if we set $n = m$, we have a two-dimensional natural density considered by Christopher (1956).

Statistical analogue of double sequences $x = (x_{jk})$ was defined as follows.

Definition 2.1: (Murasaleen & Edely, 2003): A real double sequence $x = (x_{jk})$ is statistically convergent to a number l if for each $\varepsilon > 0$, the set

$$\{(j,k), j \le n \ and \ k \le m: |x_{jk} - l| \ge \varepsilon\}$$
has double natural density zero. In this case we write $st_2 - \lim_{jk} x_{jk} = l$ and we denote the set of all statistically convergent double sequences by st_2.

Definition 2.2(Karakaya & Chisti, 2009): A sequence $x = (x_k)$ is said to be weighted statistical convergent if for every $\varepsilon > 0$,
$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \le n: P_k|x_k - L| \ge \varepsilon\}| = 0.$$

Definition 2.3 (Belen & Yildrim, 2012): A double sequence (x_{jk}) is said to be I_2 −statistical convergent to L if for each $\varepsilon > 0$ and for every $\delta > 0$,
$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \left|\{j \le m, k \le n: |x_{jk} - L| \ge \varepsilon\}\right| \ge \delta\right\} \in I_2.$$

Definition 2.4 (Alotaibi & Cakan, 2012): Let $\{\beta_m\}$and $\{\gamma_n\}$ be two sequences positive real numbers such that
$$\beta_m = \sum_{j=1}^{m} \beta_j \quad and \quad \gamma_n = \sum_{k=1}^{n} \gamma_k.$$
Then the transformation given by
$$T_{m,n} = \frac{1}{\beta_m \gamma_n} \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_j \gamma_k x_{jk}$$
is called the Riesz mean of double sequence $x = (x_{jk})$. If $\lim T_{m,n}(x) = L, L \in \mathbb{R}$, then the double sequence $x = (x_{jk})$ is said to Riesz convergent L. Then we write $S_R - lim x_{jk} = L$.

The double sequence $\theta_{r,s} = (j_r, k_s)$ is called double lacunary if there exist two increasing sequences of integers such that $j_0 = 0, h_r = j_r - j_{r-1} \to \infty$ as $r \to \infty$ and $k_0 = 0, h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. Let $j_{r,s} = j_r k_s, h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(j,k): j_{r-1} < j \le j_r \ and \ k_{s-1} < k \le k_s\}, q_r = \frac{j_r}{j_{r-1}}, \overline{q_s} = \frac{k_s}{k_{s-1}}$ and $q_{r,s} = q_r \overline{q_s}$ Savas and Patterson, 2005).

Using the notations of lacunary sequence and Riesz mean for double sequences Konca and Basarir (2016) have given a new definition:

Let $\theta_{r,s} = (j_r, k_s)$ be a double lacunary sequence and let $(\beta_j), (\gamma_k)$ be sequences of positive real numbers such that $\beta_{j_r} = \sum_{j\in(0,j_r]} \beta_j$ and $\gamma_{k_s} = \sum_{(0,k_s]} \gamma_k$. If the Riesz transformation of double sequence is RH-regular (it maps every bounded P-convergent sequence into a P-convergent sequence with the same limit),

$\theta'_{r,s} = (\beta_{j_r}, \gamma_{k_s})$ is a double lacunary sequence, that is $\beta_0 = 0, 0 < \beta_{j_{r-1}} < \beta_{j_r}$ and $H_r = \beta_{j_r} - \beta_{j_{r-1}} \to \infty$ as $r \to \infty$ and $\gamma_0 = 0, 0 < \gamma_{k_{s-1}} < \gamma_{k_s}$ and $\overline{H_r} = \gamma_{k_s} - \gamma_{k_{s-1}} \to \infty$ as $s \to \infty$.

Let $\beta_{j_{r,s}} = \beta_{j_r} \gamma_{k_s}, H_{r,s} = H_r \overline{H_s}$ and $I'_{r,s} = \{(j,k): \beta_{j_{r-1}} < j \le \beta_{j_r} \ and \ \gamma_{k_{s-1}} < k \le \gamma_{k_s}\}, Q_r = \frac{\beta_{j_r}}{\beta_{j_{r-1}}}, \overline{Q_s} = \frac{\gamma_{k_s}}{\gamma_{k_{s-1}}}$ and $Q_{r,s} = Q_r \overline{Q_s}$.

If we take $\beta_j = 1, \gamma_k = 1$ for all j and k, then $H_{r,s}, \beta_{j_{r,s}}, Q_{r,s} and \ I'_{r,s}$ reduce to $h_{r,s}, j_{r,s}, q_{r,s} and \ I_{r,s}$

Definition 2.5 (Konca, 2017): A sequence $x = (x_k)$is said to be weighted lacunary I −statistical convergent $\left(or \ \left(S_{(R,\theta)}(I) - convergent \ to \ L\right)\right)$, if for every $\varepsilon > 0$ and $\delta > 0$,
$$\left\{(r,s): \frac{1}{H_r} |\{k \in I'_r: P_k|x_k - L|\} \ge \varepsilon| \ge \delta\right\} \in I.$$

Definition 2.6 ((Konca, 2017): A sequence $x = (x_k)$ is said to be $(R, P_r, \theta)^I$ −summable to L, if $I - \lim_r \omega_r(x) \to L$ i.e., for every $\varepsilon > 0, \{r \in I: |\omega_r(x) - L| \ge \varepsilon\} \in I$ where $\omega_r(x) = 1/_{H_r} \sum_{k\in I_r} P_k x_k$. In this case we write $(R, P_r, \theta)^I - \lim x = L$ or $x_k \to L((R, P_r, \theta)^I)$.

2. MAIN RESULTS

In this section, we defined the concepts of weighted lacunary I_2 −statistical convergence of double sequence and $[R^2, \beta_m, \gamma_n, \theta_{r,s}]^{I_2}$ −summability, and examined some inclusion relations.

Definition 3.1: A double sequence $x = (x_{jk})$ is said to be weighted lacunary I_2 −statistically convergent

$\left(or\ S_{(R^2,\theta_{r,s})}(I_2) - Convergent\ to\ L\right)$, for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(r,s) \in: \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I_2.$$

In this case, we write $S_{(R^2,\theta_{r,s})}(I_2) - lim\ x_{jk} = L$ or $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$. The class of weighted lacunary I_2 −statistical convergent sequences will be denoted by $S_{(R^2,\theta_{r,s})}(I_2)$.

Definition 3.2: A double sequence $x = (x_{jk})$ is said to be $\left(R^2, \beta_m, \gamma_n, \theta_{r,s}\right)^{I_2}$ −summable to L, if $I_2 - \lim_{r,s} \sigma_{r,s}(x) \to L$ i.e for any $\varepsilon > 0, \left\{(r,s) \in I_2: \left|\sigma_{r,s}(x) - L\right| \geq \varepsilon\right\} \in I_2$ where $\sigma_{r,s}(x) = {}^1/_{H_{r,s}} \sum\sum_{j,k\in I_{r,s}} \beta_j\gamma_k\, x_{jk}$. In this case we write $\left(R^2, \beta_m, \gamma_n, \theta_{r,s}\right)^{I_2} - lim\ x_{jk} = L$ or $x_{jk} \to L\left(\left(R^2, \beta_m, \gamma_n, \theta_{r,s}\right)^{I_2}\right)$.

Theorem 3.1: Let $I_2 \subseteq P(\mathbb{N}\times\mathbb{N})$ be an admissible ideal, $\theta_{r,s}$ be a double lacunary sequence such that $I_{r,s} \subseteq I'_{r,s}$. Then $x_{jk} \to L\left(\left(R^2, \beta_{m\beta}, \gamma_n, \theta_{r,s}\right)^{I_2}\right)$ implies $x_{jk} \to L\left(S_{(R^2,\theta_{r,s}}(I_2)\right)$.

Proof: Suppose $x_{jk} \to L\left(\left(R^2, \beta_m, \gamma_n, \theta_{r,s}\right)^{I_2}\right)$ and let $K_{r,s}(\varepsilon) = \left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}$ (1.1).
Then we have

$\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \geq \frac{1}{H_{r,s}}\sum_{j,k\in I'_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \geq \frac{1}{H_{r,s}}\sum_{\substack{j,k\in I'_{r,s}\\ j,k\in K_{r,s}(\epsilon)}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right|$,
which implies

$$\frac{1}{\varepsilon}\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \geq \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right|.$$

Thus for any $\delta > 0$ we have the following

$$\left\{(r,s) \in \mathbb{N}\times\mathbb{N}: \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N}\times\mathbb{N}: \frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\delta\right\}.$$

Since $x_{jk} \to L\left(\left[R^2, \beta_m, \gamma_n, \theta_{r,s}\right]^{I_2}\right)$, it follows that the latter set belongs to I_2 and hence the result is obtained.

Theorem 3.2: Let $\beta_j\gamma_k\left|x_{jk} - L\right| \leq M$ for all$j,k \in I_2$ and $I_{r,s} \subseteq I'_{r,s}$. If $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$ then $x_{jk} \to L\left(\left[R^2, \beta_m, \gamma_n, \theta_{r,s}\right]^{I_2}\right)$.

Proof: Suppose that $\beta_j\gamma_k\left|x_{jk} - L\right| \leq M$ for all$j,k \in I_2$ and $I_{r,s} \subseteq I'_{r,s}$. If $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$ and $K_{r,s}(\varepsilon)$ be defined as in (1.1). For each $\varepsilon > 0$ we have

$$\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| \leq \frac{1}{H_{r,s}}\sum_{j,k\in I'_{r,s}}\sum \beta_j\gamma_k\left|x_{jk} - L\right|$$
$$\leq \frac{1}{H_{r,s}}\sum_{\substack{j,k\in I'_{r,s}\\ j,k\in K_{r,s}(\epsilon)}}\sum \beta_j\gamma_k\left|x_{jk} - L\right| + \frac{1}{H_{r,s}}\sum_{\substack{j,k\in I'_{r,s}\\ j,k\notin K_{r,s}(\epsilon)}}\sum \beta_j\gamma_k\left|x_{jk} - L\right|$$
$$\leq M\frac{1}{H_{r,s}}\left|K_{r,s}(\varepsilon)\right| + \varepsilon.$$

Consequently, we obtain

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}$$
$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{r,s}}\left|\left\{(j,k)\in I'_{r,s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|\leq\frac{\epsilon}{M}\right\}.$$

Since $x_{jk}\to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$, it follows that the latter set belongs to I_2, which immediately implies

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum\beta_j\gamma_k\left|x_{jk}-L\right|\right\}\in I_2.$$

This shows that $x_{jk}\to L\left(\left[R^2,\beta_m,\gamma_n,\theta_{r,s}\right]^{I_2}\right)$.

Theorem 3.3: Let $\beta_j\gamma_k\left|x_{jk}-L\right|\leq M$ for all $j,k\in I_2$ and $I_{r,s}\subseteq I'_{r,s}$. If a double sequence $x=(x_{jk})$ is $S_{(R^2,\theta_{r,s})}(I_2)$ −convergent to L, then it is $\left(R^2,\beta_m,\gamma_n,\theta_{r,s}\right)^{I_2}$ −summable to L.

Proof: Let $\beta_j\gamma_k\left|x_{jk}-L\right|\leq M$ for all $j,k\in\mathbb{N}$ and $I_{r,s}\subseteq I'_{r,s}$. Suppose that $x_{jk}\to L\left(\ S_{(R^2,\theta_{r,s})}(I_2)\right)$. Then we have the following, where $K_{r,s}(\varepsilon)$ is defined as in (3.1)

$$\left|\sigma_{r,s}-L\right|=\left|\frac{1}{H_{r,s}}\sum_{j,k\in I_{r,s}}\sum\beta_j\gamma_k\left(x_{jk}-L\right)\right|\leq\left|\frac{1}{H_{r,s}}\sum_{j,k\in I'_{r,s}}\sum\beta_j\gamma_k\left(x_{jk}-L\right)\right|$$
$$\leq\frac{1}{H_{r,s}}\sum_{j,k\in I'_{r,s}}\sum\beta_j\gamma_k\left|x_{jk}-L\right|+\frac{1}{H_{r,s}}\sum_{j,k\in I'_{r,s}}\sum\beta_j\gamma_k\left|x_{jk}-L\right|$$
$$=M\frac{1}{H_{r,s}}\left|\left\{(j,k)\in I'_{r,s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|+\varepsilon.$$

If we take $A(\varepsilon)=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{r,s}}\left|\left\{j,k\in I'_{r,s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\}$ then for $r,s\in(A(\varepsilon))^c$ if we take $\left|\sigma_{r,s}-L\right|<2\varepsilon$. Hence $\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\left|\sigma_{r,s}-L\right|\geq2\varepsilon\right\}\subset A(\varepsilon)$ and belongs to I_2. This shows that $I_2-\lim_{r,s}\sigma_{r,s}=L$, hence $x_{jk}\to L\left(\left(R^2,\beta_m,\gamma_n,\theta_{r,s}\right)^{I_2}\right)$.

Theorem 3.4: The following statements hold:

(1) If $\beta_j,\gamma_k\leq1$ for all$j,k\in I_2$ and $x_{jk}\to L\left(S_{\theta_{r,s}}(I_2)\right)$ then $x_{jk}\to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$.

(2) Let $\frac{H_{r,s}}{h_{r,s}}$ be bounded. If $\beta_j,\gamma_k\geq1$ for all $j,k\in I_2$ and $x_{jk}\to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$ then $x_{jk}\to L\left(S_{\theta_{r,s}}(I_2)\right)$.

Proof: (1) If $\beta_j,\gamma_k\leq1$ for all$j,k\in I_2$ then $H_{r,s}\leq h_{r,s}$ for all $r,s\in I_2$. So, exist M_1 and M_2 constants such that $0<M_1\leq\frac{H_{r,s}}{h_{r,s}}\leq M_2\leq1$ for all $r,s\in I_2$. Let $x_{jk}\to L\left(S_{\theta_{r,s}}(I_2)\right)$ then for an arbitrary $\varepsilon>0$ we have

$$\frac{1}{H_{r,s}}\left|\left\{(j,k)\in I'_{r,s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|$$
$$=\frac{1}{H_{r,s}}\left|\left\{\beta_{j_{r-1}}<j\leq\beta_{j_r}and\ \gamma_{k_{s-1}}<\gamma\leq\gamma_{k_s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|$$
$$\leq\frac{1}{M_1h_{r,s}}\left|\left\{\beta_{j_{r-1}}<j\leq\beta_{j_r}and\ \gamma_{k_{s-1}}<\gamma\leq\gamma_{k_s}:\beta_j\gamma_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|$$
$$=\frac{1}{M_1}\cdot\frac{1}{h_{r,s}}\left|\left\{\beta_{j_{r-1}}<j\leq\beta_{j_r}and\ \gamma_{k_{s-1}}<\gamma\leq\gamma_{k_s}:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|$$

$$= \frac{1}{M_1} \cdot \frac{1}{h_{r,s}} \left|\left\{(j,k) \in I_{r,s}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|.$$

Thus, for a given $\delta > 0$,
$\frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta \Rightarrow \frac{1}{h_{r,s}}\left|\left\{(j,k) \in I_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq M_1\delta$.
Hence,
$\left\{(r,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{(r,s) \in : \mathbb{N} \times \mathbb{N}: \frac{1}{h_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq M_1\delta\right\}$. Since $x_{jk} \to L(S_{\theta_{r,s}}(I_2))$, The set on the right hand side belongs to I_2 and so it follows that $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$.

(2) Let $\frac{H_{r,s}}{h_{r,s}}$ be upper bounded, so there exist M_1 and M_2 constants such that $1 \leq M_1 \leq \frac{H_{r,s}}{h_{r,s}} \leq M_2 < \infty$ for all $r,s \in \mathbb{N}$. If $\beta_j, \gamma_k \geq 1$ for all $j,k \in \mathbb{N}$ then dividing both side of $H_{r,s} \geq h_{r,s}$ f by $h_{r,s}$ gives $\frac{H_{r,s}}{h_{r,s}} \geq 1$ so the lower bound $M_1 \geq 1$ may indeed be taken in the bound above which justifies the lower bound assumed in the proof. If $\beta_j, \gamma_k \geq 1$ for all $j,k \in \mathbb{N}$, then we have $H_{r,s} \geq h_{r,s}$ or all $r,s \in \mathbb{N}$. Assume that $x = (x_{jk})$ converges to the limit L in $\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$,then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{h_{r,s}}\frac{1}{h_{r,s}}\left|\left\{j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\left|\left\{(j,k) \in I_{r,s}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|$$
$$= \frac{M_2}{H_{r,s}}\left|\left\{j_{r-1} \leq \beta_{j_{r-1}} < j \leq j_r \leq \beta_{j_r}, k_{s-1} \leq \gamma_{k_{s-1}} < k \leq k_s\right.\right.$$
$$\left.\left.\leq \gamma_{k_s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| = M_2\frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right|.$$

Thus, for a given $\delta > 0$,
$\frac{1}{h_{r,s}}\left|\left\{(j,k) \in I_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta \Rightarrow \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \beta_j\gamma_k\left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\delta}{M_2}$.
Hence

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{r,s}}\left|\left\{(j,k) \in I_{r,s}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subset \left\{(r,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\delta}{M_2}\right\}.$$

Since $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$. Since $x_{jk} \to L(S_{\theta_{r,s}}(I_2))$, The set on the right-hand side belongs to I_2 and so it follows that $x_{jk} \to L\left(S_{(\theta_{r,s})}(I_2)\right)$.

Theorem 3.5: For any double lacunary sequence if $\lim_{r,s} Q_{r,s} > 1$ and $x_{jk} \to L(S_{R^2}(I_2))$, then $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$.

Proof: Suppose that $\lim_{r,s} Q_{r,s} > 1$. Then there exists a $\alpha > 0$ such that $Q_{r,s} \geq 1 + \alpha$ for sufficiently large values r and s which implies that $\frac{H_{r,s}}{\beta_{j_r}\gamma_{k_s}} \geq \left(\frac{\alpha}{1+\alpha}\right)^2$. Let $x = (x_{jk}) \in S_{R^2}(I_2)$ with $S_{R^2}(I_2) - lim\ x_{jk} = L$, then for every $\varepsilon > 0$ and for sufficiently large r and s we have

$$\frac{1}{\beta_{j_r}\gamma_{k_s}}\left|\left\{j \le \beta_{j_r}, k \le \gamma_{k_s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|$$
$$\ge \frac{1}{\beta_{j_r}\gamma_{k_s}}\left|\left\{\beta_{j_{r-1}} < j \le \beta_{j_r}, \gamma_{k_{s-1}} < k \le \gamma_{k_s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|$$
$$= \frac{H_{r,s}}{\beta_{j_r}\gamma_{k_s}}\left|\left\{\beta_{j_{r-1}} < j \le \beta_{j_r}, \gamma_{k_{s-1}} < k \le \gamma_{k_s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|$$
$$\ge \left(\frac{\alpha}{1+\alpha}\right)^2 \left\{\frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|\right\} . \frac{1}{H_{r,s}}\left|\{(j,k) \in I'_{r,s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\}\right| \ge \delta \Rightarrow \frac{1}{\beta_{j_r}\gamma_{k_s}}\left|\left\{j \le \beta_{j_r}, k \le \gamma_{k_s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|$$
$$\ge \left(\frac{\alpha}{1+\alpha}\right)^2 \delta.$$

Hence

$$\left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{H_{r,s}}\left|\left\{(j,k) \in I'_{r,s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\}$$
$$\subset \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\beta_{j_r}\gamma_{k_s}}\left|\left\{j \le \beta_{j_r}, k \le \gamma_{k_s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right| \ge \left(\frac{\alpha}{1+\alpha}\right)^2 \delta\right\}.$$

Since $x_{jk} \to L(S_{R^2}(I_2))$, the set on the right-hand side belongs to I_2 and so it follows that $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$.

Theorem 3.6: Let $\theta_{r,s}$=(j_r, k_s) be a double lacunary sequence with $\limsup_{r,s} Q_{r,s} < \infty$ and $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$ then $x_{jk} \to L\big(S_{R^2}(I_2)\big)$.

Proof: If $\limsup_{r,s} Q_{r,s} < \infty$, then there is a $K > 0$ such that $Q_{r,s} \le K$ for all r, s. Let $x_{jk} \to L\left(S_{(R^2,\theta_{r,s})}(I_2)\right)$ and $N_{r,s} = \left|\left\{(j,k) \in I'_{r,s} : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right|$ (1.2)

By (1.2), given $\varepsilon > 0$ there are $r, s > r_0, s_0$. Now, let $M = max\left\{N_{r,s} : 1 \le r \le r_0 \ and \ 1 \le s \le s_0\right\}$ and let $m \ and \ n$ be any integers satisfying $j_{r-1} < m \le j_r$ and $k_{s-1} < n \le k_s$, then we can write $\frac{1}{\beta_m\gamma_n}\left|\left\{j \le \beta_m \ and \ k \le \gamma_n : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right| \le \frac{Mr_0s_0}{\beta_{j_{r-1}}\gamma_{k_{s-1}}} + \varepsilon K$. So, for a given $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\beta_m\gamma_n}\left|\left\{j \le \beta_m \ and \ k \le \gamma_n : \beta_j\gamma_k\left|x_{jk} - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\}$$
$$\subset \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{Mr_0s_0}{\beta_{j_{r-1}}\gamma_{k_{s-1}}} + \varepsilon K \ge \delta\right\}.$$

Since $\theta_{r,s}$ is a double lacunary sequence, $\beta_{j_{r-1}}\gamma_{k_{s-1}} \to \infty$ as $r, s \to \infty$, so $\frac{Mr_0s_0}{\beta_{j_{r-1}}\gamma_{k_{s-1}}} \to \varepsilon K$ as $r, s \to \infty$. Hence for the given $\delta > 0$, only finitely many pairs (r, s) can satisfy $\frac{Mr_0s_0}{\beta_{j_{r-1}}\gamma_{k_{s-1}}} + \ \varepsilon K \ge \delta$, so, the set on the right-hand side above is finite and therefore belongs to the admissible ideal I_2 since I_2 is hereditary, the set on the left-hand side, $\{(m,n) \in \ \mathbb{N}\times\mathbb{N} : \frac{1}{\beta_m\gamma_n} \ge \delta\}$, also belongs to I_2 for every $\delta > 0$. This means precisely that x_{jk} converges to L in $S_{R^2} \ of \ (I_2)$. Which completes the proof.

CONCLUSION

This paper introduced weighted lacunary I_2I2 -statistical convergence for double sequences, together with the associated weighted lacunary I_2I2-summability method $\left[R^2, \beta_m, \gamma_n, \theta_{r,s}\right]^{I_2}$[R2,βm,γn,θr,s]I2, extending to double sequences the notion of weighted lacunary II-statistical convergence previously studied for single sequences by Konca (2017).

Weighted lacunary I_2I2-summability of a double sequence was shown to imply its weighted lacunary I_2I2-statistical convergence to the same limit (Theorem 3.1); the converse holds when the weighted double sequence $\beta_j \gamma_k \mid x_{jk} - L$ |βjγk|xjk−L| is bounded (Theorems 3.2 and 3.3). Between the weighted and unweighted forms of lacunary I_2I2-statistical convergence, ordinary lacunary I_2I2-statistical convergence implies its weighted counterpart when the weight sequences (β_j)(βj) and (γ_k)(γk) are bounded above by unity, while the reverse implication holds when the weights are bounded below by unity and the ratio $H_{r,s}/h_{r,s}$Hr,s/hr,s is bounded (Theorem 3.4). Weighted lacunary I_2I2-statistical convergence and weighted (non-lacunary) I_2I2-statistical convergence coincide whenever $\liminf_{r,s} Q_{r,s} > 1$liminfr,sQr,s>1 (Theorem 3.5) or $\limsup_{r,s} Q_{r,s} < \infty$limsupr,sQr,s<∞ (Theorem 3.6).

These results recover the corresponding single-sequence results of Konca (2017) as a special case and extend the classical statistical and lacunary statistical convergence theory for double sequences to the weighted, ideal-based setting. Natural extensions of this work include weighted lacunary I_2I2-statistical convergence of order αα, weighted lacunary I_2I2-statistical Cauchy sequences, and Korovkin-type approximation theorems in this setting.

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