Research Article

On $\ell$-Fuzzy Fixed Point Results and Related Applications

Mohammed Shehu Shagari, Musa Balarabe and Kabir Lawal Yusuf
Department of Mathematics, Faculty of Physical Sciences Ahmadu Bello
University, Zaria, Nigeria
Corresponding author: shagaris@ymail.com

Keywords and Phrases:
L-Fuzzy Set,
L-Fuzzy set-valued map,
Fixed point,
$\eta$-admissible mapping,
$\eta - \psi$-contractive mapping
Metric space

ABSTRACT

Amidst several efforts in advancing fuzzy mathematics, a lot of attentions have been paid to investigate novel $\ell$-fuzzy analogues of the conventional fixed point results and their various applications. In this direction, a few new $\ell$-fuzzy fixed point theorems for $\ell$-fuzzy mappings are put forward in this paper. From application consideration, corresponding $\ell$-fuzzy fixed point results in ordered metric spaces are highlighted and analyzed. A nontrivial example is provided to validate the hypotheses of our obtained results. As some consequences, we note that the notions initiated herein complement, harmonize, and extend more than a handful of some recently proposed results in the related literature.

Corresponding author: Mohammed Shehu Shagari, Email: shagaris@ymail.com
Ahmadu Bello University, Zaria, Nigeria

INTRODUCTION

Fixed point theory is the epicenter of modern mathematical analysis with enormous applications in the study of various significant nonlinear phenomena. In metric fixed point theory, the contractive conditions on a given mapping play an important role in solving fixed point problems. The Banach contraction mapping principle (see Banach, 1922) is one of the first well-known results in metric fixed point theory. This highly celebrated result, which is an essential tool in many areas of mathematics and related disciplines, appeared in an explicit form in 1922 in Banach thesis, where it was mainly used to examine the existence of solution to an integral equation. Due to its importance and simplicity, many authors have come up with diverse extensions of the contraction mapping principle (Hussain et al. 2013; Kutbi et al., 2013; Samet et al., 2012). Along this line, (Samet et al., 2012) initiated the concepts of $\eta - \psi$ contractive and $\eta$-admissible
mappings and presented several fixed point results for such mappings. In particular, the Banach fixed point result was obtained as a consequence of the main ideas in (Samet et al., 2012).

Denote by Ξ, the class of functions ψ : R+ = [0,∞) → [0,∞) such that\[\psi^n(t) < \infty\] for all \( t > 0 \), where \( \psi^n(t) \) is the \( n \)th iterate of \( \psi \). This class of functions Ξ are known in the literature as altering distance functions. The following result is well-familiar.

**Lemma 1.1.** If \( \psi \in \Xi \), then the following hold:

(i) \( \psi^n(t)_{n\in\mathbb{N}} \) converges to 0 as \( n \to \infty \) for all \( t \in (0,\infty) \).

(ii) \( \psi(t) < t \) for all \( t > 0 \).

(iii) \( \psi(t) = 0 \) if and only if \( t = 0 \).

(Samet et al., 2012) defined the concept of \( \eta \)-admissible mappings in the following manner.

**Definition 1.2.** (Samet et al., 2012) Let \( f \) be a self-mapping on a nonempty set \( X \) and \( \eta : X^2 \to [0,\infty) \) be a function. We say that \( f \) is an \( \eta \)-admissible mapping, if

\[ x,y \in X \eta(x,y) \geq 1 \implies \eta(fx, fy) \geq 1. \]

Let \( (X,d) \) be a metric space and \( CB(X) \) denotes the set of all nonempty closed and bounded subsets of \( X \). For \( A,B \in CB(X) \), the function \( H : CB(X) \times CB(X) \to [0,\infty) \) defined by

\[ H(A, B) = \max\{ \sup d(a, B), \sup d(b, A) \}, \]

where \( d(x,A) = \inf_{x \in X} \{ d(x, a) : a \in A \} \), is called the Pompeiu-Hausdorff metric on \( CB(X) \), where \( d(x,A) = \inf_{x \in X} \{ d(x, a) : a \in A \} \).

**Remark 1.3.** It is easy to see that for all \( a,c \in X \) and \( B,C \in CB(X) \), the following are true:

(i) \( d(a,B) \leq d(a,c) + d(c,B) \).

(ii) \( d(a,B) \leq d(a,c) + H(C,B) \).

Following (Samet et al., 2012), Asl et al. (2012) introduced the notions of \( \eta_\psi \)-contractive multifunctions and \( \eta_\psi \)-admissible and proved the corresponding fixed point theorems.

**Definition 1.4.** Let \( (X,d) \) be a metric space and \( F : X \to CB(X) \) be a multivalued mapping. We say that \( F \) is an \( \eta \)-\( \psi \)-contractive multifunction, if there exist two functions \( \eta : X^2 \to \mathbb{R}^+ \) and \( \psi \in \Xi \) such that for all \( x,y \in X \),

\[ \eta(Fx, Fy) \psi(H(Fx,Fy)) \leq \psi(d(x, y)), \]

where \( \eta(A, B) = \inf \{ \eta(a, b) : a \in A, b \in B \} \).

**Definition 1.5.** (Hussain et al., 2013a) Let \( (X,d) \) be a metric space, \( F : X \to CB(X) \) be a multivalued mapping and \( \eta : X^2 \to \mathbb{R}^+ \) be a function. We say that \( F \) is an \( \eta \)-admissible whenever \( \eta(x,y) \geq 1 \) implies \( \eta(Fx, Fy) \geq 1 \).

Not long ago, (Hussain et al., 2013a) modified the ideas of \( \eta \)-admissible and \( \eta \)-\( \psi \)-contractive mappings as follows.

**Definition 1.6.** (Hussain et al., 2013a) Let \( F : CB(X) \) be a multivalued mapping, \( \eta, \theta : X^2 \to \mathbb{R}^+ \) be two functions, where \( \theta \) is bounded. We say that \( F \) is an \( \eta \)-admissible mapping with respect to \( \theta \) if for all \( x,y \in X \),

\[ \eta(x,y) \geq \theta(x,y) \implies \eta(Fx, Fy) \geq \theta(Fx,Fy), \]

where \( \theta(A, B) = \sup \{ \theta(a, b) : a \in A, b \in B \} \).

Note that if \( \theta(x,y) = 1 \) for all \( x,y \in X \), then Definition 1.6 reduces to Definition 1.5. For more recent results on \( \eta \)-\( \psi \)-contractive and \( \eta \)-admissible mappings with related fixed point theorems, the reader may consult (Ali & Kamran, 2013; Hussain et al., 2013b; Kutbi et al., 2013; Mohammed, 2021).

On the other hand, from the inception of creation, man has always been making enormous efforts in comprehending nature and then developing a strong link between life and its requirements.
launched the ideas of $D_{αL}$ and $d^{∞}_L$ distances for $L$-fuzzy sets and refined some known fixed point theorems for fuzzy and crisp set-valued mappings.

**Definition 1.7.** A relation on a nonempty set $L$ is called a partial order if it is

(i) Reflexive
(ii) Antisymmetric
(iii) Transitive.

A set $L$ together with a partial ordering is called a partially ordered set (poset, for short) and is denoted by $(L, ≤_L)$. Recall that partial orderings are used to give an order to sets that may not have a natural one.

**Definition 1.8.** Let $L$ be a nonempty set and $(L, ≤_L)$ be a partially ordered set. Then any two elements $x, y \in L$ are said to be comparable if either $x ≤_L y$ or $y ≤_L x$.

**Definition 1.9.** A partially ordered set $(L, ≤_L)$ is called:

(i) a lattice, if $x \vee y \in L$, $x \wedge y \in L$ for any $x, y \in L$;
(ii) a complete lattice, if $\forall \in L$, $\wedge \in L$ for any $\in L$;
(iii) distributive lattice if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in L$.

A partially ordered set $L$ is called a complete lattice if for every doubleton $\{x, y\}$ in $L$, either sup$\{x, y\} = x \vee y$ or inf$\{x, y\} = x \wedge y$ exists.

**Definition 1.10.** (Goguen, 1967) An $L$-fuzzy set $\nabla$ on a nonempty set $X$ is a function with domain $X$ and whose range lies in a complete distributive lattice $L$ with top and bottom elements $1_L$ and $0_L$, respectively.

Denote the class of all $L$-fuzzy sets on a nonempty set $X$ by $L^X$ (to mean a function: $X \rightarrow L$).

**Definition 1.11.** (Goguen, 1967) The $α_L$-level set of an $L$-fuzzy set $\nabla$ is denoted by $[\nabla]_{α_L}$ and is defined as follows:
\[ [\nabla]_{\alpha L} = \begin{cases} \{x \in X : 0 \leq L \nabla(x)\}, & \text{if } \alpha_L = 0 \\ \{x \in X : \alpha_L \leq L \nabla(x)\}, & \text{if } \alpha_L \in L \setminus \{0_L\} \end{cases} \]

**Definition 1.12.** (Rashid et al., 2014a; Rashid et al., 2014b) Let \( X \) be an arbitrary nonempty set and \( Y \) a metric space. A mapping \( T : X \rightarrow L^Y \) is called an \( L \)-fuzzy mapping. The function value \( T(x)(y) \) is called the degree of membership of \( y \) in \( T(x) \). For an \( L \)-fuzzy mappings \( T : X \rightarrow L^Y \), a point \( u \in X \) is called an \( L \)-fuzzy fixed point of \( T \) if there exists \( \alpha_L \in L \setminus \{0_L\} \) such that \( u \in [Tu]_{\alpha L} \).

Consistent with (Rashid et al., 2014a; Rashid et al., 2014b), we define some needed auxiliary concepts in the setting of metric space as follows. Let \((X,d)\) be a metric space and take \( \alpha_L \in L \setminus \{0_L\} \) such that \([\nabla]\alpha_L \in \Delta\alphaL \subseteq CB(X)\). Then, define
\[
p\alphaL(\nabla,\Delta) = \inf \{ d(x,y) : x \in [\nabla]\alphaL, y \in [\Delta]\alphaL \}.
\]
\[
D\alphaL(\nabla,\Delta) = H([\nabla]\alphaL, [\Delta]\alphaL). \ p(\nabla,\Delta) = \sup \alphaL p\alphaL(\nabla,\Delta).
\]
The following lemmas will be needed in the sequel.

**Lemma 1.13.** (Nadler, 1969) Let \( A \) and \( B \) be nonempty closed and bounded subsets of a metric space \((X,d)\) and \( 0 < h \in \mathbb{R} \). Then, for each \( b \in B \), there exists \( a \in A \) such that \( d(a,b) \leq H(A,B) + h \).

**Lemma 1.14.** (Ali & Kamran, 2013) Let \((X,d)\) be a metric space and \( B \) be a nonempty closed subset of \( X \) and \( q > 1 \). Then, for every \( x \in X \) with \( d(x,B) > 0 \), there exists \( b \in B \) such that \( d(x,b) \leq qd(x,B) \).

The aim of this note is to initiate a new notion of \( L \)-fuzzy set-valued maps under the name \( \eta^* - (\eta,K) \)-weak \( L \)-fuzzy contractions and investigate sufficient conditions for existence of \( L \)-fuzzy fixed points for such contractions in the framework of complete metric spaces. As some consequences of the ideas put forward herein, a few corresponding fixed point results in ordered metric space are pointed out and discussed. A nontrivial example is presented to authenticate the hypotheses of our obtained results.

**RESULTS**

Progressing along the concepts established in (Asl et al., 2012; Berinde, 2004; Hussain et al., 2013a; Hussain et al., 2014; Samet et al., 2012), we launch this section with some preliminary notions which are relevant in investigating our main \( L \)-fuzzy fixed point results. Throughout this paper, by \( L_{CB(X)} \), we mean
\[
L_{CB(X)} = \{ A \in L^X : [A]_{\alpha L} \subseteq CB(X), \text{for each } \alpha_L \in L \setminus \{0_L\} \}.
\]

**Definition 2.1.** Let \((X,d)\) be a metric space, \( T : X \rightarrow L_{CB(X)} \) be an \( L \)-fuzzy set valued map. Then \( T \) is called an \( \eta^* - \psi \)-contractive \( L \)-fuzzy set-valued map, if there exist three mappings \( \alpha_L : X \rightarrow L \setminus \{0_L\} \) and \( \eta : X^2 \rightarrow \mathbb{R}^+ \) and \( \psi \in \Xi \) such that for all \( x,y \in X \),
\[
\eta^*([Tx]_{\alpha_L}(x),[Ty]_{\alpha_L}(y))H([Tx]_{\alpha_L}(x),[Ty]_{\alpha_L}(y)) \leq \psi(d(x,y)).
\]

**Definition 2.2.** Let \((X,d)\) be a metric space, \( T : X \rightarrow L_{CB(X)} \) be an \( L \)-fuzzy set valued map. Then \( T \) is said to be \( \eta^* \)-admissible, if there exist two mappings
\[
\eta : X^2 \rightarrow \mathbb{R}^+, \ \alpha_L : X \rightarrow L \setminus \{0_L\}
\]

such that for all \( x,y \in X \),
\[
\eta(x,y) \geq 1 \text{ implies } \eta^*([Tx]_{\alpha_L}(x),[Ty]_{\alpha_L}(y)) \geq 1.
\]

**Definition 2.3.** A sequence \( \{x_n\}_{n \in \mathbb{N}} \) in a metric space \((X,d)\) is called a trajectory of an \( L \)-fuzzy set-valued map \( T : X \rightarrow L^X \), starting at \( x_1 \), if \( x_{n+1} \in [Tx_n]_{\alpha_L(x_n)} \) for each \( \alpha_L(x_n) \in L \setminus \{0_L\} \) and \( n \in \mathbb{N} \).

We denote the family of all trajectories of an \( L \) fuzzy set-valued map \( T \) at \( x \in X \) by \( T(x) \).

**Definition 2.4.** (Shukla et al., 2017) Let \((X,d)\) be a metric space and \( \eta : X^2 \rightarrow \mathbb{R}^+ \) be a mapping. A sequence \( \{x_n\}_{n \in \mathbb{N}} \) is called an \( \eta \)-sequence if \( \eta(x_n,x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \).
We denote the family of all $\eta$-sequence in $X$ by $E_{\xi \eta}$. For an $L$-fuzzy set-valued map $T : X \rightarrow LCB(X)$, we denote the $\eta$-graph of $T$ by $G_{\eta}(T)$ and is defined as:

The mapping $T$ is called an $\eta$-closed $L$-fuzzy set-valued map if $G_{\eta}(T)$ is a closed subset of $X^2$. Note that every closed mapping is a particular kind of $\eta$-closed mapping (cf. Berinde, 2004).

Throughout this paper, for all $x,y \in X$, we define $\Omega L T(x,y)$ as:

$$
\Omega L T(x,y) = \max \left\{ \frac{d(x,y)}{1 + d(x,y)} \right\}
$$

**Definition 2.6.** Let $(X,d)$ be a metric space and $T : X \rightarrow L^X$ be an $L$-fuzzy set-valued map. Then $T$ is called $(\psi,K)$-weak $L$-fuzzy contraction, if there exist $\psi \in \Xi$, $K \geq 0$ and a mapping $\alpha L : X \rightarrow L \setminus \{0_L\}$ such that for all $x,y \in X$,

$$
H([Tx]_{\alpha L}(x),[Ty]_{\alpha L}(y)) \leq \psi(\Omega L T(x,y)) + Kd(y,[Tx]_{\alpha L}(x)).
$$

(2.1)

Following (Berinde, 2004), we give the next definition which is an extension of Definition 2.6 and an $L$-fuzzy extension of the main idea in (Berinde, 2004).

**Definition 2.7.** Let $(X,d)$ be a metric space, $T : X \rightarrow L^X$ be an $L$-fuzzy set-valued map and $\eta : X^2 \rightarrow \mathbb{R}^+$ be a mapping. Then $T$ is called an $\eta - (\psi,K)$-weak $L$-fuzzy contraction, if there exist $\psi \in \Xi$, $K \geq 0$ and a mapping $\alpha L : X \rightarrow L \setminus \{0_L\}$ such that for all $x,y \in X$ and $\eta([Tx]_{\alpha L}(x),[Ty]_{\alpha L}(y)) \geq 1$,

$$
H([Tx]_{\alpha L}(x),[Ty]_{\alpha L}(y)) \leq \psi(\Omega L T(x,y)) + Kd(y,[Tx]_{\alpha L}(x)).
$$

(2.2)

**Remark 2.8.** Consistent with Berinde (Berinde, 2004), we note that due to the symmetry of the distance function, the contraction condition (2.1) implicitly includes the following dual one:

$$
H([Tx]_{\alpha L}(x),[Ty]_{\alpha L}(y)) \leq \psi(\Omega L T(x,y)) + Kd(x,[Ty]_{\alpha L}(y)).
$$

(2.3)

for all $x,y \in X$. Therefore, in order to verify $(\psi,K)$-weak $L$-fuzzy contraction of $T$, it is necessary to check both the conditions (2.2) and (2.3). However, for $\eta - (\psi,K)$weak $L$-fuzzy contraction, the inequality (2.2) is valid only for those $x,y \in X$ with $\alpha L(x),\alpha L(y) \in L \setminus \{0_L\}$ for which $\eta([Tx]_{\alpha L}(x),[Ty]_{\alpha L}(y)) \geq 1$. The advantage here is that it is not necessary to verify condition (2.2) for all $x,y \in X$.

Now, we state and prove our main $L$-fuzzy fixed point result as follows.

**Theorem 2.9.** Let $(X,d)$ be an $\eta$-complete metric space and $T : X \rightarrow L^X$ be an $\eta - (\psi,K)$-weak $L$-fuzzy contraction. Suppose further that the following conditions are satisfied:

$$
G_{\eta}(T) = \{ (x_n,x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_{n \in \mathbb{N} } \in E_{\xi \eta} \cap T_{(x_1)} \}.
$$
$(C_1)$ for each $x \in X$, there exists $a_L(x) \in L \setminus \{0_L\}$ such that $[T_x]_{a_L(x)}$ is a nonempty closed and bounded subset of $X$;

$(C_2)$ $T$ is $\eta$-admissible;

$(C_3)$ there exists $x_0 \in X$ and $x_1 \in [T_{x_0}]_{a_L(x_0)}$ with $\eta(x_0, x_1) \geq 1$;

$(C_4)$ for a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in $E_{\mathbb{X}_0}$, starting at $x_1$ and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then, there exists $u \in X$ and $a_L(u) \in L \setminus \{0_L\}$ such that $u \in [T_u]_{a_L(u)}$.

**Proof.** By Condition $(C_3)$, there exist $x_0 \in X$ and $x_1 \in [T_{x_0}]_{a_L(x_0)}$ such that $\eta(x_0, x_1) \geq 1$. If $x_0 = x_1$, the proof is finished. We presume that $x_0 \neq x_1$. If there exists $x_1 \in [T_{x_0}]_{a_L(x_1)}$ for some $a_L(x_1) \in L \setminus \{0_L\}$, then $x_1$ is an $L$-fuzzy fixed point of $T$. So suppose that $x_1 \not\in [T_{x_0}]_{a_L(x_1)}$. Then, given that $T$ is $\eta$-admissible and $\eta(x_0, x_1) \geq 1$, we have $\eta^*([T_{x_0}]_{a_L(x_0)}, [T_{x_1}]_{a_L(x_1)}) \geq 1$. It follows that

$$d(x_1, [T_{x_1}]_{a_L(x_1)}) \leq H([T_{x_0}]_{a_L(x_0)}, [T_{x_1}]_{a_L(x_1)})$$

(2.4) $\leq \psi(\Omega T(x_0, x_1)) + Kd(x_1, [T_{x_0}]_{a_L(x_0)})$,

where

$$\Omega T(x_0, x_1) = \max \left\{ d(x_0, x_1), d(x_0, [T_{x_0}]_{a_L(x_0)}), d(x_1, [T_{x_1}]_{a_L(x_1)}), \right.$$  

$$d(x_0, [T_{x_1}]_{a_L(x_1)})d(x_1, [T_{x_0}]_{a_L(x_0)}), \frac{d(x_0, [T_{x_0}]_{a_L(x_0)})d(x_1, [T_{x_1}]_{a_L(x_1)})}{1 + d(x_0, x_1)} \right\}$$

$$\leq \left\{ d(x_0, x_1), d(x_0, x_1), d(x_1, [T_{x_1}]_{a_L(x_1)}), \right.$$  

$$\frac{d(x_0, x_1)d(x_1, [T_{x_1}]_{a_L(x_1)})}{1 + d(x_0, x_1)} \right\}$$

$$\leq \max \{d(x_0, x_1), d(x_1, [T_{x_1}]_{a_L(x_1)})\}.$$  

If $\max \{d(x_0, x_1), d(x_1, [T_{x_1}]_{a_L(x_1)})\} = d(x_1, [T_{x_1}]_{a_L(x_1)})$, then by Lemma 1.1, it follows from (2.4) that

$$d(x_1, [T_{x_1}]_{a_L(x_1)}) \leq \psi(d(x_1, [T_{x_1}]_{a_L(x_1)})) < d(x_1, [T_{x_1}]_{a_L(x_1)})$$

a contradiction. Hence,

$$\max \{d(x_0, x_1), d(x_1, [T_{x_1}]_{a_L(x_1)})\} = d(x_0, x_1).$$

Consequently,

$$d(x_1, [T_{x_1}]_{a_L(x_1)}) \leq \psi(d(x_0, x_1)).$$

(2.5)

For any $y > 1$, by Lemma 1.14, there exists $x_2 \in [T_{x_1}]_{a_L(x_1)}$ such that

$$d(x_1, x_2) \leq yd(x_1, [T_{x_1}]_{a_L(x_1)}) \leq y\psi(d(x_0, x_1)).$$

Clearly, if $x_1 = x_2$, then $T$ has at least one $L$-fuzzy fixed point in $X$ and the proof is complete.

Suppose that $x_1 \neq x_2$ so that $d(x_1, x_2) > 0$. Take $y_1 = \frac{\psi(\psi(yd(x_0, x_1)))}{\psi(d(x_0, x_1))}. Then,

$$y_1 > 1.$$  

Since $x_1 \in [T_{x_0}]_{a_L(x_0)}, x_2 \in [T_{x_1}]_{a_L(x_1)}$, and

$$\eta^*([T_{x_0}]_{a_L(x_0)}, [T_{x_1}]_{a_L(x_1)}) \geq 1,$$

we get

$$\eta(x_1, x_2) \geq 1.$$ And $\eta^*$-admissibility of $T$ implies that $\eta^*([T_{x_1}]_{a_L(x_1)}, [T_{x_2}]_{a_L(x_2)}) \geq 1.$ If $x_2 \in [T_{x_2}]_{a_L(x_2)}$ for some $a_L(x_2) \in L \setminus \{0_L\}$, then $x_2$ is an $L$-fuzzy fixed point of $T$. So, let $x_2 \in [T_{x_2}]_{a_L(x_2)}$, then

$$d(x_2, [T_{x_2}]_{a_L(x_2)}) \leq H([T_{x_1}]_{a_L(x_1)}, [T_{x_2}]_{a_L(x_2)})$$

(2.6) $\leq \psi(\Omega T(x_1, x_2)) + Kd(x_2, [T_{x_1}]_{a_L(x_1)})$,

where
\[ \Omega_L T(x_1, x_2) = \max \left\{ d(x_1, x_2), d(x_1, [T x_1]_{\alpha_L(x_1)}), d(x_2, [T x_2]_{\alpha_L(x_2)}), \right. \\
\left. \frac{d(x_1, [T x_2]_{\alpha_L(x_2)}) - d(x_2, [T x_1]_{\alpha_L(x_1)})}{1 + d(x_1, x_2)} \right\} \]
\[ \leq \max \left\{ d(x_1, x_2), d(x_1, x_2), d(x_2, [T x_2]_{\alpha_L(x_2)}), \right. \\
\left. \frac{d(x_1, x_2) - d(x_2, [T x_2]_{\alpha_L(x_2)})}{1 + d(x_1, x_2)} \right\} \]

Assume that \( \max\{d(x_1, x_2), d(x_2, [T x_2]_{\alpha_L(x_2)})\} = d(x_2, [T x_2]_{\alpha_L(x_2)}) \), then by Lemma 0.1, (6) yields
\[ d(x_2, [T x_2]_{\alpha_L(x_2)}) \leq \psi(d(x_2, [T x_2]_{\alpha_L(x_2)})) < d(x_2, [T x_2]_{\alpha_L(x_2)}), \]
a contradiction. Therefore, \( \max\{d(x_1, x_2), d(x_2, [T x_2]_{\alpha_L(x_2)})\} = d(x_1, x_2) \). It follows that
\[ d(x_2, [T x_2]_{\alpha_L(x_2)}) \leq \psi(d(x_1, x_2)). \] (0.7)

Since \( \omega_1 > 1 \), then by Lemma 0.14, there exists \( x_3 \in [T x_2]_{\alpha_L(x_2)} \) such that
\[ d(x_2, x_3) \leq \omega_1 d(x_2, [T x_2]_{\alpha_L(x_2)}) \leq \omega_1 \psi(d(x_1, x_2)) = \psi(\omega_1 \psi(d(x_0, x_1))). \]

By continuing in this fashion, we construct a sequence \( \{x_n\}_{n \in \mathbb{N}} \) with \( x_n \in [T x_{n-1}]_{\alpha_L(x_{n-1})} \) \( x_n \neq x_{n-1} \), \( \eta(x_n, x_{n-1}) \geq 1 \) for all \( n \in \mathbb{N} \) such that
\[ d(x_n, x_{n-1}) \leq \psi^{n-1}(\omega_1 \psi(d(x_0, x_1))). \] (0.8)

Next, we shall show that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). For this, let \( m, n \in \mathbb{N} \) with \( m > n \), then
\[ d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(\omega_1 \psi(d(x_0, x_1))). \] (0.9)

Since \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for all \( t > 0 \), it follows from (0.9) that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). Therefore, \( \{x_n\}_{n \in \mathbb{N}} \in E_{X \eta} \cap T_r(x_0) \) is a Cauchy sequence. The \( \eta \)-completeness of this space implies that there exists \( u \in X \) such that \( x_n \longrightarrow u \) as \( n \longrightarrow \infty \).

To show that \( u \) is an \( L \)-fuzzy fixed point of \( T \), assume contrary that \( u \notin [T u]_{\alpha_L(u)} \) with \( d(u, [T u]_{\alpha_L(u)}) > 0 \). Now, since \( \{x_n\}_{n \in \mathbb{N}} \in E_{X \eta} \cap T_r(x_0) \) and \( x_n \longrightarrow u \) as \( n \longrightarrow \infty \),
it follows from \((C_4)\) that \(\eta(x_n, u) \geq 1\) for each \(n \in \mathbb{N}\). Moreover,
\[
d(u, [Tu]_{\alpha_L(u)}) \leq d(u, x_{n+1}) + d(x_{n+1}, [Tu]_{\alpha_L(u)}) \\
\leq d(u, x_{n+1}) + H([Tx_n]_{\alpha_L(x_n)}, [Tu]_{\alpha_L(u)}) \\
\leq \psi(\Omega_T(x_n, u)) + Kd(u, [Tx_n]_{\alpha_L(x_n)}) + d(x_{n+1}, u) \\
\leq \psi(\Omega_T(x_n, u)) + Kd(u, x_{n+1}) + d(x_{n+1}, u),
\]
(0.10)

where
\[
\Omega_T(x_n, u) = \max \left\{d(x_n, u), d(x_n, [Tx_n]_{\alpha_L(x_n)}), d(u, [Tu]_{\alpha_L(u)}) \right\} \\
\leq \max \left\{d(x_n, u), d(x_n, x_{n+1}), d(u, [Tu]_{\alpha_L(u)}) \right\} \\
+ \frac{d(x_n, x_{n+1})d(u, [Tu]_{\alpha_L(u)})}{1 + d(x_n, u)}
\]

Taking limit in (0.10) as \(n \to \infty\), there exists \(n_0 \in \mathbb{N}\) such that \(\Omega_T(x_n, u) = d(u, [Tu]_{\alpha_L(u)})\) as \(n \to \infty\) for all \(n \geq n_0\). Hence, Lemma 0.1 implies that
\[
d(u, [Tu]_{\alpha_L(u)}) \leq \psi(d(u, [Tu]_{\alpha_L(u)})) < d(u, [Tu]_{\alpha_L(u)}),
\]

is a contradiction. Consequently, there exists \(u \in X\) and \(\alpha_L(u) \in L \setminus \{0_L\}\) such that \(u \in [Tu]_{\alpha_L(u)}\); that is, \(u\) is an \(L\)-fuzzy fixed point of \(T\).

**Corollary 0.24.** Let \((X, d)\) be an \(\eta\)-complete metric space and \(T : X \to L^X\) be an \(L\)-fuzzy set-valued map. Suppose further that the following conditions are satisfied:

\((C_1)\) for each \(x \in X\), there exists \(\alpha_L(x) \in L \setminus \{0_L\}\) such that \([Tx]_{\alpha_L(x)} \in CB(X)\);

\((C_2)\) there exists \(\gamma \in (0, 1)\) and \(K \geq 0\) such that for all \(x, y \in X\),
\[
H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) \leq \gamma d(x, y) + Kd(y, [Tx]_{\alpha_L(x)});
\]

\((C_3)\) \(T\) is \(\eta\)-admissible;

\((C_4)\) there exist \(x_0 \in X\) and \(x_1 \in [Tx_0]_{\alpha_L(x_0)}\) with \(\eta(x_0, x_1) \geq 1\);

\((C_5)\) for a trajectory \(\{x_n\}_{n \in \mathbb{N}}\) in \(E_{X, \eta}\), starting at \(x_1\) and converging to \(x \in X\), we have \(\eta(x_n, x) \geq 1\) for all \(n \in \mathbb{N}\).

Then, there exist \(u \in X\) and \(\alpha_L(u) \in L \setminus \{0_L\}\) such that \(u \in [Tu]_{\alpha_L(u)}\).

**Proof.** Take \(\psi(t) = \gamma t\) for all \(t \geq 0\) in Theorem 0.23.
The next result in association with $d^\eta_\alpha$-metric for $L$-fuzzy sets extends the ideas presented in (Azam et al., 2010; Heilpern, 1981; Park & Jeong, 1997). It is pertinent to note that $L$-fuzzy fixed-point results in the setting of $d_\infty$-metric is very useful in computing Hausdorff dimensions. These dimensions help us to understand the concepts of $e^\infty$-space which is of great importance in higher energy physics (see, e.g. (El-Naschi, 2002; El-Naschi, 2000).

For all $x,y \in X$, we define $\Omega_L(x,y,p)$ as:

$$\Omega_L(x,y,p) = \max \left\{ d(x,y), p(x,T(x)), p(y,T(y)), \frac{p(x,T(x))p(y,T(y))}{1+d(x,y)} \right\}. $$

**Theorem 2.11.** Let $(X,d)$ be an $\eta$-complete metric space and $T : X \rightarrow W(X)$ be an $L$-fuzzy set-valued map. Assume that the following conditions are satisfied:

(C1) there exists $K \geq 0$ such that for all $x,y \in X$,

$$d^\eta_\alpha(Tx,Ty) \leq \psi(\Omega_L(x,y,p)) + Kp(y,T(x));$$

(2.11)

(C2) $T$ is $\eta$-admissible;

(C3) there exists $x_0 \in X$ and $x_1 \in T(x_0)$ with $\eta(x_0,x_1) \geq 1$;

(C4) for a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in $E_{\eta\phi}$ starting at $x_1$ and converging to $x \in X$, we have $\eta(x_0,x) \geq 1$ for all $n \in \mathbb{N}$.

Then, $T$ has an $L$-fuzzy fixed point in $X$.

**Proof.** Let $x \in X$, by assumption, $[Tx]_L$ is a nonempty closed and bounded subset of $X$. Now, for all $x,y \in X$,

$$D_{1L}(T(x),T(y)) \leq d_{LL}^\eta(T(x),T(y)) \leq \psi(\Omega_T(x,y,p)) + Kp(y,T(x)).$$

Since $[Tx]_L \subseteq [Tx]_{\alpha L(\alpha)} \in CB(X)$ for each $\alpha L(\alpha) \in L \setminus \{0_L\}$, therefore, $d(x,[Tx]_{\alpha L(\alpha)}) \leq d(x,[Tx]_L)$ for each $\alpha L(\alpha) \in L \setminus \{0_L\}$. This implies that $p(x,T(x)) \leq d(x,[Tx]_L)$. On similar arguments, $p(y,T(x)) \leq d(y,[Tx]_L)$.

Hence,

$$H([Tx]_L,[Ty]_L) \leq D_{1L}(T(x),T(y)) \leq \psi(\Omega_L(T(x,y,p)) + Kp(T(x)) \leq \psi(\Omega_L(T(x,y)) + Kd(y,[Tx]_L)).$$

Hence, Theorem 2.9 can be employed to find $u \in X$ and $\alpha L(u) = 1_L$ such that $u \in [Tu]_L$.

In the next result, we replace Condition (C4) of Theorem 2.9 with $\eta$-closedness of the mapping $T$.

**Theorem 2.12.** Let $(X,d)$ be an $\eta$-complete metric space and $T : X \rightarrow L^X$ be an $\eta$-(\$\psi,K$)-weak $L$-fuzzy contraction.

Assume further that the following conditions are satisfied:

(C1) for each $x \in X$, there exists $\alpha L(x) \in (0,1]$ such that $[Tx]_{\alpha L(x)} \in CB(X)$;

(C2) $T$ is $\eta$-admissible;

(C3) there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha L(\alpha)}$ with $\eta(x_0,x_1) \geq 1$;

(C4) $T$ is an $\eta$-closed fuzzy set-valued map.

Then, there exist $u \in X$ and $\alpha L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{\alpha L(u)}$.

**Proof.** In line with the proof of Theorem 2.9, we have a sequence $\{x_n\}_{n \in \mathbb{N}} \in E_{\eta\phi} \setminus T(x_0)$ such that $x_n \rightarrow u \in X$ as $n \rightarrow \infty$.

To see that $u$ is an $L$-fuzzy fixed point of $T$, first notice that since $\{x_n\}_{n \in \mathbb{N}} \in E_{\eta\phi} \cap$
Let \( T \) be a mapping. For each \( x \in X \), consider an \( L \)-valued map, the \( \eta \)-graph \( G_\eta(T) \) is a complete distributive lattice. Let \( X \) be a complete metric space. Given that \( T \) is an \( \eta \)-closed \( L \)-fuzzy set-valued map, the \( \eta \)-graph \( G_\eta(T) \) is a closed subset of \( X^2 \). Thus, letting \( n \to \infty \) and noting that \( x_n \to u \) as \( n \to \infty \), we have

\[
\lim (x_{n-1},x_n) \in G_\eta(T) \text{ implies } (u,u) \in G_\eta(T).
\]

By definition of \( G_\eta(T) \), we obtain some \( u \in X \) and \( \alpha_L(u) \in L \setminus \{0_L\} \) such that \( u \in [Tu]_{\alpha_L(u)} \).

In what follows, we provide an example to support the hypotheses of Theorem 2.9.

**Example 2.13.** Let \( L = \{a,b,c,g,s,m,n,v\} \) be such that \( a \preceq_L s \preceq_L c \preceq_L v \), \( a \preceq_L g \preceq_L b \preceq_L v \), \( s \preceq_L m \preceq_L v \), \( g \preceq_L m \preceq_L v \), \( n \preceq_L v \); and each elements of the doubletons \{c,m\}, \{m,b\}, \{s,n\}, \{n,g\} are not comparable. It follows that \((L, \preceq_L)\) is a complete distributive lattice. Let \( X = [0,\infty) \) and \( d : X^2 \to \mathbb{R}^+ \) be defined by

\[
d(x,y) = \begin{cases} 
0, & \text{if } x = y \\
\max\{x,y\}, & \text{if } x \neq y.
\end{cases}
\]

Then, \((X,d)\) is an \( \eta \)-complete metric space. Let \( \alpha_L : X \to L \setminus \{0\} \) be a mapping. For each \( x \in X \), consider an \( L \)-fuzzy set-valued map \( T(x) : X \to L \) defined as follows: if \( x = 0 \), then

\[
T(x)(t) = \begin{cases} 
v, & \text{if } t = 0 \\
a, & \text{if } t \neq 0,
\end{cases}
\]

if \( x \in (0,4) \), then

\[
T(x)(t) = \begin{cases} 
v, & \text{if } 0 \leq t < \frac{1}{56} \\
^3, & \text{if } \frac{1}{56} \leq t < \frac{1}{5} \\
a, & \text{if } \frac{1}{5} \leq t < \frac{1}{6} \\
a, & \text{if } \frac{1}{6} \leq t < \infty.
\end{cases}
\]

if \( x > 4 \), then

\[
T(x)(t) = \begin{cases} 
v, & \text{if } 0 \leq t < \frac{1}{56} \\
b, & \text{if } \frac{1}{56} \leq t < \frac{1}{5} \\
c, & \text{if } \frac{1}{5} \leq t < \frac{1}{6} \\
a, & \text{if } \frac{1}{6} \leq t < \infty.
\end{cases}
\]

Suppose that \( \alpha_L(x) = v \) for each \( x \in X \), then

\[
[T_x]_{\alpha_L(x)} = \begin{cases} 
\{0\}, & \text{if } x = 0 \\
\left[0, \frac{1}{5}\right], & \text{if } x \in (0,4] \\
\left\{\frac{1}{6}\right\}, & \text{if } x > 4.
\end{cases}
\]

Obviously, \([Tx]_{\alpha_L(x)} \subset CB(X)\) for each \( x \in X \).

Define the mappings \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \eta : X^2 \to \mathbb{R}^+ \) respectively as follows:

\[
\psi(t) = \begin{cases} 
0, & \text{if } t \in [0,4] \\
\frac{1}{5}, & \text{if } t > 4.
\end{cases}
\]

\[
\eta(x,y) = \begin{cases} 
3, & \text{if } x, y \in [0,4] \\
\max\{x,y\}, & \text{if } x, y \in (4,\infty).
\end{cases}
\]

Now, we verify Conditions \((C_2),(C_3)\) and \((C_4)\) of Theorem 2.9. Let \( x, y \in (0,4] \) and \( \eta(x,y) \geq 1 \). Then, \( \eta*([Tx]_{\alpha_L(x)},[Ty]_{\alpha_L(y)}) = \inf\{\eta(p,q) : p \in [Tx]_{\alpha_L(x)}, q \in [Ty]_{\alpha_L(y)}\} = 3 \)

that is, \( T \) is \( \eta \)-admissible. Hence, \((C_2)\) holds good. Take \( x_0 \in (0,4) \), then \([Tx_0]_{\alpha_L(x_0)} = \{0, \frac{1}{17}\} \) if \( x_1 \in [Tx_0]_{\alpha_L(x_0)} \), then \( x_1 \in \left[0, \frac{1}{17}\right] \subset [0,4] \) and \( \eta(x_0,x_1) = 3 \). Thus, \((C_3)\) is satisfied.

Now, consider a sequence \( \{x_n\}_{n \in \mathbb{N}} \) with \( \eta(x_n,x_{n+1}) \geq 1 \) such that \( \lim_{n \to \infty} x_n = x \in X \). Then, we are sure that \( x_n \in [0,4] \). It follows that \( x \in [0,4] \). Therefore, \( \eta(x_n,x) \geq 1 \) for all \( n \in \mathbb{N} \), proving \((C_4)\). To see that \( T \) is an \( \eta \)-\((\psi, K)\) weak \( L \)-fuzzy contraction, first observe that for \( x = y = 0 \) or \( x, y \in (4,\infty) \), there is nothing to show, since

\[
H([Tx]_{\alpha_L(x)},[Ty]_{\alpha_L(y)}) = H([0], [0]) = 0 = H\left(\left\{\frac{1}{6}\right\}, \left\{\frac{1}{6}\right\}\right).
\]
So, if $\eta(x,y) \geq 1$ and $x, y \in [0,4]$ with $x \neq y$, we consider the following cases: Case I. If $x < y$, then $H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = H(0, \frac{y}{17}) = \frac{y}{17}$, $d(x,y) = y, d([Tx]_{\alpha_L(x)} = x, d(y, [Ty]_{\alpha_L(y)}) = y, d(y, [Tx]_{\alpha_L(x)}) = y$, and

$$d(x, [Ty]_{\alpha_L(y)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{17} \\ x, & \text{if } x > \frac{y}{17} \\ y, & \text{if } x > \frac{y}{17}. \end{cases}$$

So, for each $K \geq \frac{1}{9}$ it is easy to check that the contraction Condition (2.2) is satisfied.

Case II. If $x, y \in [0,4]$ and $x > y$, then $H([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}) = H(0, \frac{y}{17}) = \frac{y}{17}$, $d(x,y) = x, d(x, [Tx]_{\alpha_L(x)}) = x$, $d(y, [Ty]_{\alpha_L(y)}) = y$, and

$$d(y, [Tx]_{\alpha_L(y)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{17} \\ y, & \text{if } x > \frac{y}{17}. \end{cases}$$

Thus, the inequality (2.2) holds for all $K \geq \frac{1}{9}$. It follows that $T$ is an $\eta, \psi, K$-weak $L$-fuzzy contraction. Therefore, all the hypotheses of Theorem 2.9 are satisfied. Hence, there exists $u = 0 \in X$ and $\alpha_L(0) = v$ such that $0 \in [T0]_{\alpha_L(0)}$.

**Consequences in ordered metric spaces**

The study of existence of fixed points on metric spaces endowed with a partial order is one of the very interesting improvements in the area of fixed point theory. This trend was introduced by (Turinici, 1986) in 1986, but it became one of the core research subjects after the publications of the results of (Ran & Reurings, 2004) and (Nieto & Rodriguez, 2005).

In this section, we study the analogue of our main result in the setting of ordered metric spaces. Some preliminary concepts, introduced as follows, are necessary. Accordingly, $(X, d, \preceq)$ is called an ordered metric space, if:

(i) $(X,d)$ is a metric space, and

(ii) $(X, \preceq)$ is a partially ordered set (poset).

**Definition 3.1.** Let $(X, \preceq)$ be a poset, $A, B \subseteq X$ and $T : X \longrightarrow L^X$ be an $L$-fuzzy set-valued map. Then, we write $A \vee B$ if an only if $a \preceq b$ for all $a \in A$ and $b \in B$. The mapping $T$ is called order preserving if for each $x, y \in X$, there exist $a_L(x), a_L(y) \in L \setminus \{0_L\}$ such that $x \preceq y$ implies $[Tx]_{a_L(x)} \vee [Ty]_{a_L(y)}$.

**Definition 3.2.** Let $(X, d, \preceq)$ be an ordered metric space. An $L$-fuzzy set-valued map $T : X \longrightarrow L^X$ is called an ordered $(\psi, K)$-weak $L$-fuzzy contraction, if for all $x, y \in X$, there exist $a_L(x), a_L(y) \in L \setminus \{0_L\}$ with $[Tx]_{a_L(x)} \vee [Ty]_{a_L(y)} \in \Xi$ and $K \geq 0$ such that

$$H([Tx]_{a_L(x)}, [Ty]_{a_L(y)}) \leq \psi(\Omega_L(T(x,y))) + Kd(x, [Tx]_{a_L(x)}).$$

(3.1)

Consistent with (Shukla, 2017), we say that a sequence $(x_n)_{n \in \mathbb{N}}$ is called an ordered sequence if $x_n \preceq x_{n-1}$ for all $n \in \mathbb{N}$. The class of all ordered sequences in $X$ is denoted by $E_X^{\preceq}$. For an $L$-fuzzy set-valued map $T : X \longrightarrow L^X$, we denote by $G^\preceq(T)$, the $\gamma$-graph of $T$, where

$$G^\preceq(T) = \{(x_n, x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_{n \in \mathbb{N}} \in E_X^{\preceq} \cap T_{r(x_1)}(x_1)\}.$$  

The $L$-fuzzy set-valued map $T$ is said to be closed, if $G^\preceq(T)$ is a closed subset of $X^2$. The metric space $(X, d, \preceq)$ is called $\gamma$-complete, if every Cauchy sequence in $E_X^{\preceq}$ is convergent in $X^2$.

The following result is an extension of the main results of Ran and Reurings (2004) and Nieto and Rodriguez (2005) into $L$-fuzzy domain.

**Corollary 3.3.** Let $(X, d, \preceq)$ be an $\gamma$-complete metric space and $T : X \longrightarrow L^X$ be an ordered $(\psi, K)$-weak $L$-fuzzy...
contraction. Assume that the following conditions are satisfied:

(C1) for each $x \in X$, there exists $a_L(x) \in L \setminus \{0_L\}$ such that $[Tx]_{a_L(x)} \in CB(X)$;

(C2) $T$ is order preserving;

(C3) there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{a_L(x_1)}$ with $x_1 \preceq x_1$;

(C4) at least one of the following hypotheses holds:

(i) for a trajectory $\{x_n\}_{n \in \mathbb{N}} \in E_X \ast$ starting at $x_1$ and converging to $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;

(ii) $T$ is an $\mathbb{L}$-closed $\mathbb{L}$-fuzzy set-valued map.

Then, there exists $u \in X$ and $a_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{a_L(u)}$. Proof. Consider a mapping $\eta : X^2 \rightarrow \mathbb{R}^+$ defined by

$$\eta(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$$

If $\eta(x, y) \geq 1$, then, by order preserveness of $T$, $[Tx]_{a_L(x)} \cap [Ty]_{a_L(y)}$ for some $a_L(x), a_L(y) \in (0, 1]$. Hence,

$$\eta^*(\{[Tx]_{a_L(x)} \cap [Ty]_{a_L(y)}\}) = \inf\{\eta(p, q) : p \in [Tx]_{a_L(x)}, q \in [Ty]_{a_L(y)}\} = 1.$$

This implies that $T$ is an $\eta$-admissible $\mathbb{L}$-fuzzy set-valued map. Moreover, from Condition (C3), there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{a_L(x_1)}$ such that $\eta(x_0, x_1) \geq 1$. And, by (i) of Condition (C4), for any trajectory $\{x_n\}_{n \in \mathbb{N}} \in E_X \ast$ converging to $x \in X$, we must have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. In addition, it is clear that $\mathbb{L}$-closedness of $T$ implies the $\eta$-closedness of $T$ and $\mathbb{L}$-completeness of $(X, d, \preceq)$ yields $\eta$-completeness. Since $T$ is an ordered $(\psi, K)$-weak $\mathbb{L}$-fuzzy contraction, by definition of $\eta$, the mapping $T$ is an $\eta$-$(\psi, K)$-weak $\mathbb{L}$-fuzzy contraction. Consequently, Theorem 2.9 or 2.12 can be availed to obtain $u \in X$ and some $a_L(u) \in L \setminus \{0_L\}$ such that $u \in [Tu]_{a_L(u)}$.

**CONCLUSION**

Based on the ideas of $\mathbb{L}$-fuzzy sets, a new notion of $\mathbb{L}$-fuzzy set-valued maps under the name $\eta$-$(\psi, K)$-weak $\mathbb{L}$-fuzzy contractions is initiated and sufficient criteria for existence of $\mathbb{L}$-fuzzy fixed points for such contractions in the framework of complete metric spaces are investigated in this note. As some consequences of the obtained results herein, a few related fixed point theorems in ordered metric spaces are underlined and analyzed. An example is constructed to support the hypotheses of our established results. It is observed conclusively that the concepts proposed in this work harmonize, improve and complement some recently published results regarding $\eta$-$\psi$-contractive mappings and their corresponding fixed point theorems.

**Competing Interests**

The author declares that there is no competing interests.

**Acknowledgement**

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that helped to improve this manuscript.
REFERENCES


