



Research Article

Sine Burr XII Distribution: Properties and Application to Real Data Sets

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ABSTRACT

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New parameters can be added to expand families of distribution for greater flexibility or to construct covariate models in several ways. In this paper, a trigonometric-type distribution was developed to generate a flexible distribution without adding additional parameters taking the Burr XII distribution as the base distribution. The new distribution is called the Sine-Burr XII distribution. Some of the mathematical properties of this distribution such as the moment, moment generating function, entropy and order statistics were derived. The parameters of the new distribution were estimated using the Maximum Likelihood method. The applicability of the Sine-Burr XII distribution was illustrated using two real datasets.

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INTRODUCTION

The accuracy of parametric statistical inference and modelling of data sets largely depends on how well the probability distribution fits the given data sets once it has met all the necessary distributional

assumptions. Many pieces of research have been conducted on statistical distribution in search to generate distribution with more desirable and adjustable properties that can be

models real-life data sets of the varying form of density and failure rate functions.

Presently most studies are focused on developing new families that are a generalisation of existing distribution to obtain a better fit for the modelling of data. These families of distribution are obtained by either compounding two or more distributions or adding one or more parameters to the baseline model. Many writers have substantially reviewed the various families of distribution (Hamedani *et al.*, 2018).

The main reason for developing this family is to improve the flexibility of the existing classical distribution, thus enabling them to provide a better fit to the survival data set than other distributions with the same number of parameters and model different kinds of failure rates (Monotonic and non-monotonic).

The Burr system of distribution comprises 12 types of cumulative distribution functions

which give a variety of density forms. The desirability of this relatively unknown family of distribution for model fitting is that it combines a simple mathematical expression for cumulative frequency function with coverage in the skewness-kurtosis plane. Many standard theoretical distributions including the Weibull, exponential, logistic, generalized logistic, Gompertz, normal extreme value and uniform distribution are special cases of limiting cases of the Burr types system of distribution.

The family of Burr type distribution is a very well-known distribution family for modelling survival data and for modelling phenomena with monotone and unimodal failure rates. In most areas of application, the Burr distribution of any type is adequate and as such various generalizations and extensions of the Burr, distributions have been suggested in the statistical literature.

The cumulative density function for a random variable \mathbf{X} having a Burr XII distribution is given by:

$$H(x) = 1 - \frac{1}{(1 + x^a)^b} \quad (1)$$

The probability density function (pdf) of a Burr XII distribution for the random variable \mathbf{X} is also given by:

$$h(x) = \frac{abx^{a-1}}{(1 + x^a)^{b+1}}, x, a, b \quad (2)$$

The trigonometric transformations provide versatility and flexibility as the parameter(s) oscillate with change in its value and the periodic function controls the way the distribution curve behaves. The composition of various functions with different behaviour provides a better make for modelling real-world phenomena which the previous established generalized statistical distributions lacked. With the suitable composition of

trigonometric functions and generalized distributions by considering the parameters suiting to data to be modelled, the shortcoming is minimized or better said the distribution becomes apt to model data of different nature and gives rise to even newer transformed distribution to work with.

Some of the recently developed trigonometric distributions include an exponentiated sine-

generated family of distributions by Muhammad *et al.* (2021), Sin-G class of distributions by Souza *et al.* (2019), sine square distribution by Al-Faris and Khan (2008), Sine Inverse Lomax Generated Family

by Fayomi *et al.* (2021), Sine Kumaraswamy-G family of distributions by Chinesu and Jamal (2021), Sine Topp-Leone family by El-Babtain *et al.* (2020) and Sine Power Lomax distribution by Nagarjuna *et al.* (2021).

METHODOLOGY

probability density function (pdf), then, the Sine-G family of probability distribution according to Kumar *et al.* (2015) is given by:

Sine G Family of Probability Distribution

Let $H(x)$ be the cumulative distribution function (cdf) of a univariate continuous distribution and $h(x)$ be the corresponding

$$F(x, \xi) = \int_0^{\frac{\pi}{2}H(x, \xi)} \cos t \, dt = \sin \left\{ \frac{\pi}{2} H(x, \xi) \right\} \tag{3}$$

and its corresponding pdf is given by:

$$f(x, \xi) = \frac{\pi}{2} h(x, \xi) \cos \left\{ \frac{\pi}{2} H(x, \xi) \right\} \tag{4}$$

where $H(x, \xi)$ and $h(x, \xi)$ are the cdf and the pdf of any baseline distribution with vector parameter ξ .

The New Sine Burr XII Distribution

The pdf of the newly developed Sine-Burr XII distribution is given by:

$$f(x; a, b) = \frac{\pi}{2} \frac{abx^{a-1}}{(1+x^a)^{b+1}} \cos \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1+x^a)^b} \right] \right\} \tag{5}$$

and the corresponding cdf is given by:

$$F(x; a, b) = \sin \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1+x^a)^b} \right] \right\}, \quad a, b, x > 0 \tag{6}$$

The survival function $S(x)$, hazard function $h(x)$, the reverse hazard function $r(x)$, cumulative hazard function $H(x)$ and the quantile function of the Sine Burr XII are given below:

This function is derived by inverting the cumulative density function (cdf) of any given continuous probability distribution. It is used for obtaining some moments like skewness and kurtosis as well as the median and for the generation of random variables from the distribution in question (Hyndman & Fan 1996). The survival function is given by

Survival Function

$$S(x; a, b) = 1 - F(x; a, b)$$

$$S(x; a, b) = 1 - \sin \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1+x^a)^b} \right] \right\} \tag{7}$$

Hazard Function

The hazard rate function is a significant quantitative exploration of life phenomenon. The hazard rate function measures the conditional instantaneous rate of failure at time x , given survival to time x . Barlow *et al.*, (1963) introduced hazard rate function, in studying the relationship between properties of a distribution function (or

$$h(x; a, b) = \frac{f(x; a, b)}{s(x; a, b)} = \frac{f(x; a, b)}{1 - F(x; a, b)}$$

where $f(x; a, b)$ and $F(x; a, b)$ are the *pdf* and *cdf* of the Sine-Burr XII distribution given in equation (5) and (6) respectively.

$$h(x; a, b) = \frac{\frac{\pi}{2} \frac{abx^{a-1}}{(1+x^a)^{b+1}} \cos\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\}}{1 - \sin\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\}} \tag{8}$$

The reversed hazard rate of the random variable X , $r(x; a, b)$, according to Veres-Ferrer and Pavía (2014) is the quotient between the density and distribution functions:

$$r(x; a, b) = \frac{f(x; a, b)}{F(x; a, b)} = \frac{d}{dx}(\ln F(x; a, b))$$

For the Sine-Burr XII distribution, the reverse hazard function is given by:

$$r(x; a, b) = \frac{\frac{\pi}{2} \frac{abx^{a-1}}{(1+x^a)^{b+1}} \cos\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\}}{\sin\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\}}$$

$$r(x; a, b) = \frac{\pi}{2} \frac{abx^{a-1}}{(1+x^a)^{b+1}} \cot\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\} \tag{9}$$

The cumulative hazard function ($H(x; a, b)$) is given by:

$$H(x; a, b) = -\ln\left\{1 - \sin\left[\frac{\pi}{2}\left(1 - \frac{1}{(1+x^a)^b}\right)\right]\right\} \tag{10}$$

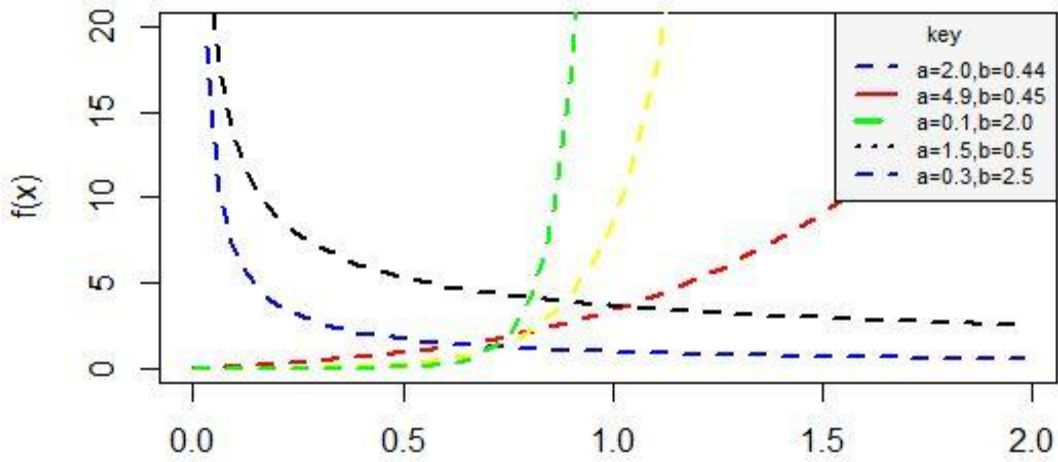
To partition probability distributions, the quantile function $Q(u)$ is used. The median of a distribution can also be obtained and random samples of various sizes can be simulated. It is possible to obtain the Quantile function by taking the inverse of a distribution's cumulative distribution function.

$$Q(u) = F^{-1}(x)$$

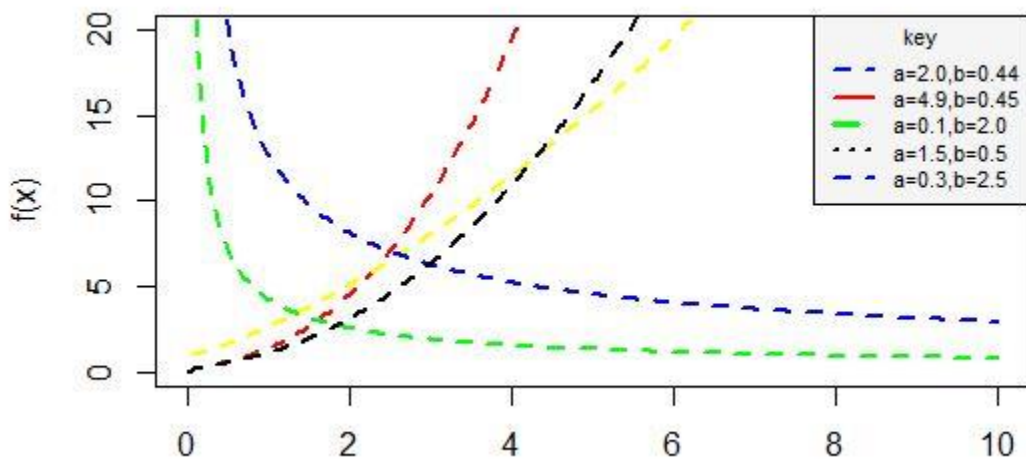
$$Q(u) = F^{-1}\left\{\left[1 - \frac{2 \sin^{-1} u}{\pi}\right]^{-\frac{1}{b+1}} - 1\right\}^{\frac{1}{a}} \tag{11}$$

The Plots of the pdf, hazard rate function are given below:

density function) and corresponding hazard rate function. It has significant value in the study of reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology. The hazard function which is denoted by $h(x; a, b)$ is given as



Plot of the pdf of Sine-Burr XII distribution



Hazard rate function of the Sine-Burr XII distribution

Useful Expansion

$$f(x; a, b) = \frac{\pi}{2} \frac{abx^{a-1}}{(1+x^a)^{b+1}} \cos\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\}$$

we first of all expand

$$(1+x^a)^{-(b+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{b+i}{i} x^{ai}$$

$$\cos\left\{\frac{\pi}{2}\left[1 - \frac{1}{(1+x^a)^b}\right]\right\} = \sum_{j=0}^{\infty} \frac{(-1)^j \pi^{2j}}{(2i)! 2^{2j}} \left[1 - \frac{1}{(1+x^a)^b}\right]^{2j}$$

$$\left[1 - \frac{1}{(1+x^a)^b}\right]^{2j} = \sum_{k=0}^{\infty} (-1)^k \binom{2j}{k} (1+x^a)^{-k(b+1)}$$

$$(1 + x^a)^{-k(b+1)} = \sum_{l=0}^{\infty} (-1)^l \binom{k(b+1) + l - 1}{l} x^{al}$$

Therefore,

$$f(x; a, b) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{ab(-1)^{i+j+k+l} \pi^{2j+1}}{(2j)! 2^{2j+1}} \binom{b+i}{i} \binom{2j}{k} \binom{k(b+1) + l - 1}{l} x^{a-1+al+ai}$$

$$\text{Let } \Phi = \frac{ab(-1)^{i+j+k+l} \pi^{2j+1}}{(2j)! 2^{2j+1}} \binom{b+i}{i} \binom{2j}{k} \binom{k(b+1) + l - 1}{l}$$

Therefore,

$$f(x; a, b) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi x^{a(i+l+1)-1} \tag{12}$$

$$F(x; a, b) = \sin \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1+x^a)^b} \right] \right\} = \sum_{p=0}^{\infty} \frac{(-1)^p \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \left[1 - \frac{1}{(1+x^a)^b} \right]^{2p+1}$$

$$\left[1 - \frac{1}{(1+x^a)^b} \right]^{2p+1} = \sum_{q=0}^{\infty} (-1)^q \binom{2p+1}{q} (1+x^a)^{-q(b+1)}$$

$$(1+x^a)^{-q(b+1)} = \sum_{s=0}^{\infty} (-1)^s \binom{q(b+1) + s - 1}{s} x^{as}$$

$$F(x; a, b) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{p+q+s} \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \binom{2p+1}{q} \binom{q(b+1) + s - 1}{s} x^{as}$$

$$\text{Let } \Theta = \frac{(-1)^{p+q+s} \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \binom{2p+1}{q} \binom{q(b+1) + s - 1}{s}$$

Then the cdf will be expressed as:

$$F(x; a, b) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \Theta x^{as} \tag{13}$$

Mathematical Properties

The mathematical properties of the Sine Burr XII distribution such as the rth moment,

moment generating function, entropy and order statistics were derived and presented as follows:

Moment

moment of the random variable X with probability density function $f(x)$ is given by

In this subsection, we discuss the r^{th} Moment for triangular-G Family of distribution. The r^{th}

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x; a, b) dx = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi \int_0^{\infty} x^{r+a(i+l+1)-1} dx$$

$$\text{Let } \psi = \int_0^{\infty} x^{r+a(i+l+1)-1}$$

Therefore, the r th moment of the Sine-Burr XII distribution is given by:

$$\begin{aligned} \mu'_r &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi \psi \end{aligned} \tag{14}$$

Moment Generating Function

The moment generating function of a random variable X is the expected value of e^{tx} . We say that the moment generating function exist if there exists a positive constant a such that $M_x(t)$ is finite for $\forall t \in [-a, a]$.

$$\begin{aligned} M_x(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x; a, b) dx \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi \int_0^{\infty} e^{tx} x^{a(i+l+1)-1} dx \end{aligned}$$

$$\text{Let } Y = \int_0^{\infty} e^{tx} x^{a(i+l+1)-1} dx$$

Therefore, the Moment Generating Function of the Sine-Burr XII distribution is given by:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi Y \tag{15}$$

Entropy

Entropy is used as a measure of information or uncertainty, which is present in a random observation of its actual population. There will be greater uncertainty in the data if the value of entropy is large. The Shannon entropy for the true continuous random variable X is defined as:

$$I_{\lambda}(x) = \frac{1}{1-\lambda} \log \int_{-\infty}^{\infty} f(x; a, b)^{\lambda} dx$$

$$f(x; a, b)^\lambda = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi x^{a(i+l+1)-1} \right)^\lambda$$

$$f(x; a, b)^\lambda = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi \right)^\lambda (x^{a(i+l+1)-1})^\lambda$$

Let $\beta = (x^{a(i+l+1)-1})^\lambda$

Then,

$$I_\lambda(x) = \frac{1}{1-\lambda} \left[\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Phi \right)^\lambda \log \int_{-\infty}^{\infty} \beta^\lambda dx \right] \tag{16}$$

Order Statistics

Let X_1, X_2, \dots, X_n be a random sample of size n from a continuous population having a pdf $f(x)$ and distribution function (cdf) $F(x)$, Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics (OS). David (1970) defined the pdf of $X_{r:n}$ that is the r^{th} OS by:

$$f_{r:n}(x; a, b) = \frac{\left(\sum_{i,j,k,l=0}^{\infty} \Phi x^{a(i+l+1)-1} \right)}{B(r, n-r+1)} \left(\sum_{p,q,s=0}^{\infty} \Theta x^{as} \right)^{r-1} \left(1 - \left(\sum_{p,q,s=0}^{\infty} \Theta x^{as} \right) \right)^{n-r} \tag{17}$$

Parameter Estimation

In this paper, we explore the maximum likelihood technique to estimate the unknown parameters of the Sine-Burr XII distribution for complete data. Maximum likelihood estimates (MLEs) have appealing qualities that may be used to generate confidence ranges

and provide simple approximations that function well in finite samples. In distribution theory, the resulting approximation for MLEs is easily handled, either analytically or numerically. Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the Sine-Burr XII distribution. Then, the likelihood function of the Sine-Burr XII is given as follows:

$$l(\varphi) = n \log \left(\frac{\pi}{2} \right) + n \log a + n \log b + (a-1) \sum_{i=0}^n \log x - (b+1) \sum_{i=0}^n (1+x^a) + \sum_{i=0}^n \log \cos \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1+x^a)^b} \right] \right\}$$

Differentiating $l(\varphi)$ with respect of a gives the following expression:

$$\frac{\partial l(\varphi)}{\partial a} = \frac{n}{a} + \sum_{i=0}^n \log x - (b + 1) \sum_{i=0}^n \frac{x^a \log a}{(1 + x^a)} + \sum_{i=0}^n \frac{\pi}{2} \frac{bx^a \log a}{(1 + x^a)^{b+1}} \tan \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1 + x^a)^b} \right] \right\} \quad (18)$$

Differentiating $l(\varphi)$ with respect to b gives the following expression:

$$\frac{\partial l(\varphi)}{\partial a} = \frac{n}{b} - \sum_{i=0}^n \log(1 + x^a) - \sum_{i=0}^n \frac{\log(1 + x^a)}{(1 + x^a)} \tan \left\{ \frac{\pi}{2} \left[1 - \frac{1}{(1 + x^a)^b} \right] \right\} \quad (19)$$

Equation (18) and equation (19) gives the maximum likelihood estimators of the parameters a and b of the Sine-Burr XII distribution.

RESULTS AND DISCUSSION

Application

Two datasets were considered for illustrative purposes and comparison with Burr XII (BXII) and Extended Burr XII distributions. For each data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method. We obtain the values of the Akaike information criterion (AIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC).

First Dataset

We consider Vinyl Chloride data of sample thirty-four obtained from clean up gradient monitoring wells in mg/L; this data set was used by Bhaumik et al. (2009). The data are as follows: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1,

8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2

Second Dataset

The second dataset consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul (Hinkley, 1977). The data are as follows: 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05

Table 1 and table 2 gives the summary statistics of the two data sets such as the mean, the median, the first and third quartile, the minimum and the maximum values.

Table 1: Summary statistics of the two datasets

Data	Minimum	Q ₁	Media	Mean	Q ₃	Maximum
Dataset I	0.100	0.500	1.150	1.879	2.550	8.00
Dataset II	0.320	0.915	1.470	1.675	2.087	4.75

Table 2: MLE, AIC, CAIC, BIC, and HQIC of the two data sets

Data set I	<i>a</i>	<i>b</i>	<i>MLE</i>	<i>AIC</i>	<i>CAIC</i>	<i>BIC</i>	<i>HQIC</i>
SBXII	13.648	1.722	22.469	48.939	49.326	51.992	49.980
BXII	0.397	5.007	26.549	57.099	57.487	60.152	58.141
ExBXII	1.346	1.384	55.702	115.404	57.487	118.456	116.445
Dataset II	<i>a</i>	<i>b</i>	<i>MLE</i>	<i>AIC</i>	<i>CAIC</i>	<i>BIC</i>	<i>HQIC</i>
SBXII	0.500	0.500	18.685	41.369	41.814	44.172	42.266
BXII	6.939	3.712	25.618	55.236	55.680	58.038	56.133
ExBXII	2.868	0.881	39.187	82.373	55.680	85.175	83.269

Table 2 gives the results of the analysis of the two datasets. The results of the analysis of the Sine-Burr XII were compared with Burr XII distribution and Extended Burr XII distribution. The proposed Sine- Burr XII distribution has proven to be the better model because it has the least AIC, CAIC, BIC and HQIC. It can be seen that the proposed model outperformed its competitors with minimum AIC, BIC, CAIC and HQIA.

CONCLUSION

There has been a growing interest among statisticians and applied researchers in developing flexible lifetime models for the betterment of modelling survival data. In this paper, we introduced a two-parameter Sine-Burr XII distribution which is obtained by considering a Burr XII distribution as the baseline. We study some of its statistical and mathematical properties. Maximum Likelihood Estimation was used in parameter estimation. The usefulness of the new distribution was illustrated via the analysis of two real data sets. We hope that the proposed extended model will attract wider applications in survival analysis.

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