



Research Article

Sine Type II Topp-Leone G Family of Probability
Distribution: Mathematical Properties and Application

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ABSTRACT

In this paper we introduced a new trigonometric family of continuous distributions called the Sine Type II Topp-Leone-G family of distributions. The new proposed family is more flexible than some of the existing families. The models of the family have many applications in physics, engineering, and medicine. Some statistical properties of the Sine Type II Topp-Leone-G family of distributions such as moments, moment generating function, quantile function and order statistics are derived. Maximum likelihood estimation method was used to estimate parameters of the model. Two real life datasets were analyzed to show the flexibility of the proposed model and the results of the analysis showed that, the proposed model was more efficient and best fit the data sets than its competitors.

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INTRODUCTION

The statistical literature contains many new classes of distributions which have been constructed by extending common families of continuous distributions. These new families have been used for modeling data in many areas such as engineering, economics, biological studies, environmental sciences. The general objectives of generalizing a new family of distributions include the following: produce skewness for symmetrical models; define special models with different shapes of hazard rate function; construct heavy-tailed distributions for modeling various real data sets; make the kurtosis more flexible compared to that of the baseline distribution; generate distributions which are skewed, J-shaped or reversed-J shaped and provide consistently better fits than other generalized distributions with the same underlying model (Yousof *et al.*, 2018).

Recently, some attempts have been made to define new trigonometric distributions to extend well known distributions and at the same time provide greater flexibility in modeling data in practice. Some of the recently developed trigonometric families of distributions include: Exponentiated Sine-G by

$$F(x; \xi) = \int_0^{\frac{\pi}{2} H(x; \xi)} \cos t dt = \sin \left\{ \frac{\pi}{2} H(x; \xi) \right\} \quad (1)$$

and its corresponding pdf is given by:

$$f(x; \xi) = \frac{\pi}{2} h(x; \xi) \cos \left\{ \frac{\pi}{2} H(x; \xi) \right\} \quad (2)$$

where $H(x; \xi)$ and $h(x; \xi)$ are the cdf and the pdf of any baseline distribution with vector parameter ξ .

Type II Topp-Leone-G Family

The cdf of the Type II Topp-Leone G family of distribution according to Elgarhy *et al.*, (2018) is defined by:

$$F(x; \alpha, \xi) = 1 - [1 - G^2(x; \xi)]^\alpha \quad (3)$$

with corresponding pdf defined by:

$$f(x; \alpha, \xi) = 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \quad (4)$$

where α is a shape parameter, $g(x; \xi)$ and $G(x; \xi)$ are pdf and cdf of any baseline distribution respectively and ξ is a vector parameter of the baseline distribution.

Muhammad *et al.*, (2021), Arctan X family of distribution by Alkhairy *et al.*, (2021), The Tan-G class by Souza *et al.*, (2021), Arcsine Exponentiated-G by He *et al.*, (2020), Sine Inverse Lomax-G family by Famoye *et al.*, (2021), Sine Topp Leone-G by Al Babtain (2020), Tangent Topp-Leone G Family by Nanga *et al.*, (2022), Sine-G family by Sakthivel and Rajkumar (2021), Sine Alpha Power G Family by Benchiha *et al.*, (2023) and Sine Extended Frechet G family by Jamal *et al.*, (2021). These distributions are commonly applied to describe real world phenomena. Due to the usefulness of statistical distributions from these families, their theory is widely studied and new distributions were developed. This research article adopted a trigonometric transformation in developing new flexible family of distributions without adding additional parameters.

MATERIALS AND METHODS

The Sine-G Family

Let $H(x; \xi)$ be the cumulative distribution function (cdf) of a Univariate continuous distribution and $h(x; \xi)$ be the corresponding probability density function (pdf), then, the Sine-G family of probability distribution according to Kumar *et al.*, (2015) is given by:

The Proposed Sine Type II Topp-Leone-G Family

The cdf of the new Sine Type II Topp-Leone G Family is given by:

$$F(x; \alpha, \xi) = \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\} \tag{5}$$

with pdf given by:

$$f(x; \alpha, \xi) = \frac{\pi}{2} 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\} \tag{6}$$

Survival Function

The survival function which gives the probability that a patient will survive longer

$$S(x; \alpha, \xi) = 1 - \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\} \tag{7}$$

Hazard Function

The hazard rate function measures the conditional instantaneous rate of failure at time x , given survival to time x . Barlow *et al.*, (1963) introduced hazard rate function, in studying the relationship between properties of a distribution function (or density function) and corresponding hazard

than specified period of time t , for the survival function of the Sine Type II Topp-Leone-G Family of Distributions is given by:

rate function. It has significant value in the study of reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology. The hazard function of the Sine Type II Topp-Leone-G family $h(x; \alpha, \xi)$ is given by:

$$h(x; \alpha, \xi) = \frac{\frac{\pi}{2} 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\}}{1 - \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\}} \tag{8}$$

Reverse Hazard Function

The reversed hazard rate of a random life is defined as the ratio between the life probability density to its distribution

$$f(x; \alpha, \xi) = \frac{\pi}{2} 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \cot \left\{ \frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right\} \tag{9}$$

function. The hazard rate function of the Sine Type II Topp-Leone-G family of distribution is given by:

Cumulative Hazard Function

The cumulative hazard function is a function which provides the total accumulated risk of experiencing the event of interest that has been

gained by progressing to time t . The cumulative hazard rate function of the Sine Type II Topp-Leone G family is given by:

$$H(x; \alpha, \xi) = -\ln \left\{ 1 - \sin \left[\frac{\pi}{2} \left[1 - \left(1 - G^2(x; \xi) \right)^\alpha \right] \right] \right\} \tag{10}$$

Quantile Function

To partition probability distributions, the quantile function $Q(u)$ is used. The median of a distribution can also be obtained and random samples of various sizes can be simulated. It is possible to obtain the Quantile function by

taking the inverse of a distribution's cumulative distribution function. The quantile function of the Sine Type II Topp-Leone G family is given by:

$$Q(u) = G^{-1} \left\{ 1 - \left[1 - \frac{\sin^{-1} u}{\pi/2} \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{2}}$$

Therefore, the quantile function of the TIITL-L distribution is given by:

$$x = \Phi(u) = G^{-1} \left[1 - \left(1 - \frac{\sin^{-1}(u)}{\pi/2} \right)^{\frac{1}{\alpha}} \right]^{\frac{1}{2}} \tag{11}$$

where $G^{-1}(x, \xi)$ is the quantile function of the baseline distribution $G(x; \xi)$.

The pdf and the cdf of the Sine Type II Topp-Leone G family can be expanded using power series expansion as follows:

Mixture Representation

$$f(x; \alpha, \xi) = \frac{\pi}{2} 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \cot \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\}$$

$$[1 - G^2(x; \xi)]^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} G^{2i}(x; \xi)$$

$$\cos \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\} = \sum_{j=1}^n \frac{(-1)^j \pi^{2j}}{(2j)! 2^{2j}} \left[1 - (1 - G^2(x; \xi))^\alpha \right]^{2j}$$

$$\left[1 - (1 - G^2(x; \xi))^\alpha \right]^{2j} = \sum_{k=0}^{\infty} (-1)^k \binom{2j}{k} (1 - G^2(x; \xi))^{2j\alpha}$$

$$(1 - G^2(x; \xi))^{2j\alpha} = \sum_{l=0}^{\infty} (-1)^l \binom{2j\alpha}{l} G^{2l}(x; \xi)$$

Therefore, the pdf can be expressed as:

$$f(x; \alpha, \xi) = \sum_{i,j,k,l=1}^n \frac{(-1)^{i+j+k+l} \pi^{2j+1}}{(2j)! 2^{2j+1}} \binom{\alpha-1}{i} \binom{2j}{k} \binom{2j\alpha}{l} g(x; \xi) G(x; \xi)^{2(i+j)+1}$$

Let $\Psi = \frac{2\alpha(-1)^{i+j+k+l} \pi^{2j+1}}{(2j)! 2^{2j+1}} \binom{\alpha-1}{i} \binom{2j}{k} \binom{2j\alpha}{l}$

Therefore,

$$f(x; \alpha, \xi) = \sum_{i,j,k,l=1}^n \Psi g(x; \xi) G(x; \xi)^{2(i+j)+1}$$

(12)

The cdf can also be expanded as follows:

$$\sin \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\} = \sum_{p=0}^{\infty} \frac{(-1)^p \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \left[1 - (1 - G^2(x; \xi))^\alpha \right]^{2p+1}$$

$$\left[1 - (1 - G^2(x; \xi))^\alpha \right]^{2p+1} = \sum_{q=0}^{\infty} (-1)^q \binom{2p+1}{q} (1 - G^2(x; \xi))^{\alpha(2p+1)}$$

$$(1 - G^2(x; \xi))^{\alpha(2p+1)} = \sum_{r=0}^{\infty} (-1)^r \binom{\alpha(2p+1)}{r} G^{2r}(x; \xi)^{\alpha(2p+1)}$$

Therefore, the cdf is given as:

$$F(x; \alpha, \xi) = \sum_{p,q,r=0}^{\infty} \frac{(-1)^{p+q+r} \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \binom{2p+1}{q} \binom{\alpha(2p+1)}{r} G(x; \xi)^{2\alpha(2p+1)}$$

$$\text{Let } \Phi = \frac{(-1)^{p+q+r} \pi^{2p+1}}{(2p+1)! 2^{2p+1}} \binom{2p+1}{q} \binom{\alpha(2p+1)}{r}$$

Therefore,

$$F(x; \alpha, \xi) = \sum_{p,q,r=0}^{\infty} \Phi G(x; \xi)^{2\alpha(2p+1)} \tag{13}$$

The expression in equation (12) and equation (13) are the reduced forms of the pdf and the cdf of the new Sine Type II Topp-Leone G family of distribution.

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x; \alpha, \xi) dx$$

$$\mu_r' = \sum_{i,j,k,l=1}^{\infty} \Psi \int_0^{\infty} x^r g(x; \xi) G(x; \xi)^{2(i+j)+1} dx$$

Let μ_r'

Therefore,

$$\mu_r' = \sum_{i,j,k,l=1}^{\infty} \Psi \varpi \tag{14}$$

Moment Generating Function

The moment generating function of a random variable X is defined as $E(e^{tx})$. We say that mgf of X exists, if there exists a positive

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x; \alpha, \xi) dx$$

$$M_x(t) = \sum_{i,j,k,l=1}^n \Psi \int_0^{\infty} e^{tx} g(x; \xi) G(x; \xi)^{2(i+j)+1} dx$$

$$\text{Let } \beta = \int_0^{\infty} e^{tx} g(x; \xi) G(x; \xi)^{2(i+j)+1} dx$$

$$M_x(t) = \sum_{i,j,k,l=1}^{\infty} \Psi \beta \tag{15}$$

Entropy

Entropy is used as a measure of information or uncertainty, which present in a random observation of its actual population. There will be the greater uncertainty in the data if the

$$I_{\theta}(x) = \frac{1}{1-\theta} \log \int_{-\infty}^{\infty} f(x; \alpha, \xi)^{\theta} dx$$

Mathematics Properties

Moment

Moments is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis. The r^{th} moments of the Sine Type II Topp Leone G family of distribution is obtained as follow:

constant a such that $M_t(x)$ is finite for all $s \in [-a, a]$. The moment generating function of the Sine Type II Topp-Leone G family is given by:

value of entropy is large. For some probability distributions expression, the differential entropy is considered mostly effective. The entropy for the Sine Type II Topp-Leone G family is given by:

$$f(x; \alpha, \xi)^\theta = \left(\sum_{i,j,k,l=1}^n \Psi g(x; \xi) G(x; \xi)^{2(i+j)+1} \right)^\theta$$

$$f(x; \alpha, \xi)^\theta = \left(\sum_{i,j,k,l=1}^n \Psi \right)^\theta \left(g(x; \xi) G(x; \xi)^{2(i+j)+1} \right)^\theta$$

Let $\lambda = g(x; \alpha, \xi) G(x; \alpha, \xi)^{2(i+j)+1}$

Therefore,

$$I_\theta(x) = \frac{1}{1-\theta} \left[\left(\sum_{i,j,k,l=1}^n \Psi \right)^\theta \log \int_{-\infty}^{\infty} \lambda^\theta dx \right] \tag{16}$$

Equation (16) gives the entropy of the proposed Sine Type II Topp-Leone G family of distribution.

Order Statistics

Let x_1, x_2, \dots, x_n be a random sample of size n from a continuous population having a pdf $f(x)$

$$f_{r:n}(x) = \frac{f(x)}{B(m, n-m+1)} (F(x))^{m-1} (1-F(x))^{n-m}$$

The order statistics of the Sine Type II Topp-Leone is given by:

$$f_{r:n}(x) = \frac{\sum_{i,j,k,l=0}^{\infty} \psi g(x; \xi) G(x; \xi)^{2(i+j)+1}}{B(m, n-m+1)} \left(\sum_{p,q,r=0}^{\infty} \Phi G(x; \xi)^{2\alpha(2p+1)} \right)^{m-1} \left(1 - \sum_{p,q,r=0}^{\infty} \Phi G(x; \xi)^{2\alpha(2p+1)} \right)^{n-m} \tag{17}$$

Parameter Estimation

Maximum Likelihood Estimate

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the Sine Type II Topp-Leone G

$$f(x; \alpha, \xi) = \frac{\pi}{2} 2\alpha g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{\alpha-1} \cot \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\}$$

$$\begin{aligned} \ell &= n \log \left(\frac{\pi}{2} \right) + n \log 2 + n \log \alpha + \sum_{i=1}^n \log g(x; \xi) + \sum_{i=1}^n \log G(x; \xi) + (\alpha - 1) \sum_{i=1}^n \log (1 - G^2(x; \xi)) \\ &+ \sum_{i=1}^n \log \cos \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\} \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n (1 - G^2(x; \alpha, \xi)) + \frac{\pi}{2} (1 - G^2(x; \alpha, \xi))^\alpha \ln (1 - G^2(x; \alpha, \xi)) \\ &\times \tan \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\} \end{aligned} \tag{19}$$

and cdf $F(x)$, Let $X_{1:n} \leq X \leq X_{n:n}$ be the corresponding order statistics (OS). David (1970) defined the pdf of $X_{1:n}$ that is the i^{th} order statistics by:

family of distribution. Then, the likelihood function of the Sine Type II Topp-Leone G family is derived as follows:

$$\frac{\partial \ell}{\partial \xi} = \sum_{i=0}^n \frac{g'(x; \xi)}{g(x; \xi)} + \sum_{i=0}^n \frac{g(x; \xi)}{G(x; \xi)} + (\alpha - 1) \sum_{i=1}^n \frac{2g(x; \xi)G(x; \xi)}{[1 - G(x; \xi)]} + \sum_{i=1}^n \pi \alpha g(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{\alpha-1} \times \tan \left\{ \frac{\pi}{2} \left[1 - (1 - G^2(x; \xi))^\alpha \right] \right\} \tag{20}$$

Equation (19) gives the maximum likelihood estimate of the parameter α while equation (20) is the maximum likelihood parameter estimate of the vector parameter ξ of any baseline distribution

Sub-Model

Sine Type II Topp-Leone Lomax Distribution

Suppose the baseline distribution is Lomax with pdf and cdf given by:

$$f(x; \theta, \lambda) = \frac{\theta}{\lambda} \left(1 + \frac{x}{\lambda} \right)^{-(\theta+1)} \text{ and } F(x; \theta, \lambda) = 1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta}$$

Where θ is the scale parameter while λ is the shape parameter. Then, the pdf of the Sine Type II Topp-Leone-Lomax distribution is given by

$$f(x; \alpha, \theta, \lambda) = \frac{\pi}{2} 2\alpha \frac{\theta}{\lambda} \left(1 + \frac{x}{\lambda} \right)^{-(\theta+1)} \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right) \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^{\alpha-1} \cos \left\{ \frac{\pi}{2} \left[1 - \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^\alpha \right] \right\}$$

and its corresponding cdf is given by

$$F(x; \alpha, \theta, \lambda) = \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right)^\alpha \right] \right\}$$

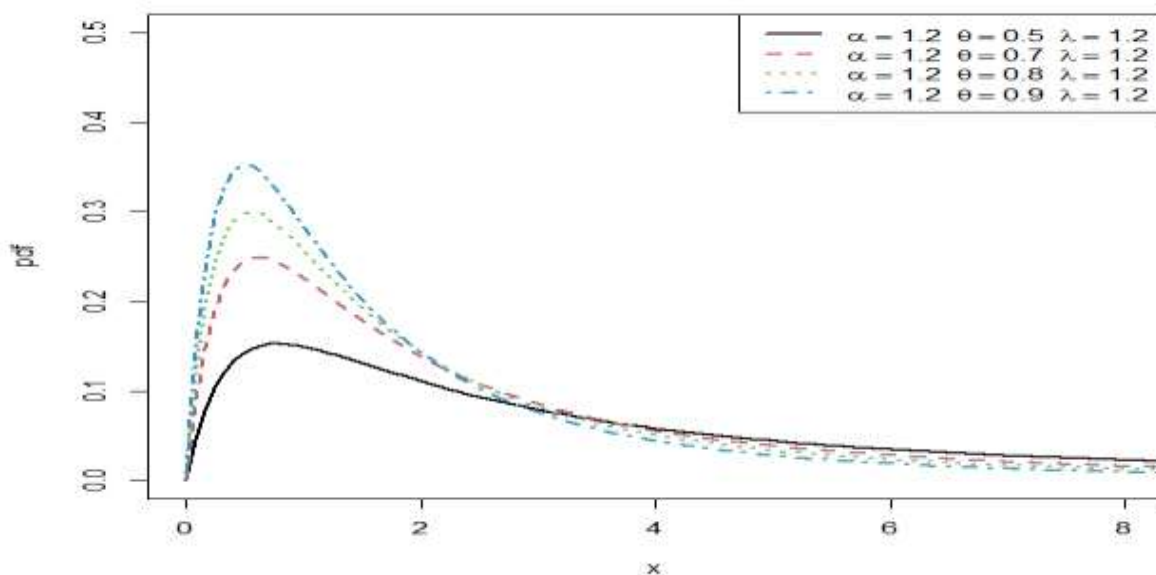


Figure 1: Plot of the pdf of Sine Type II Topp-Leone-Lomax distribution for varying value of θ and fixed values of α and λ

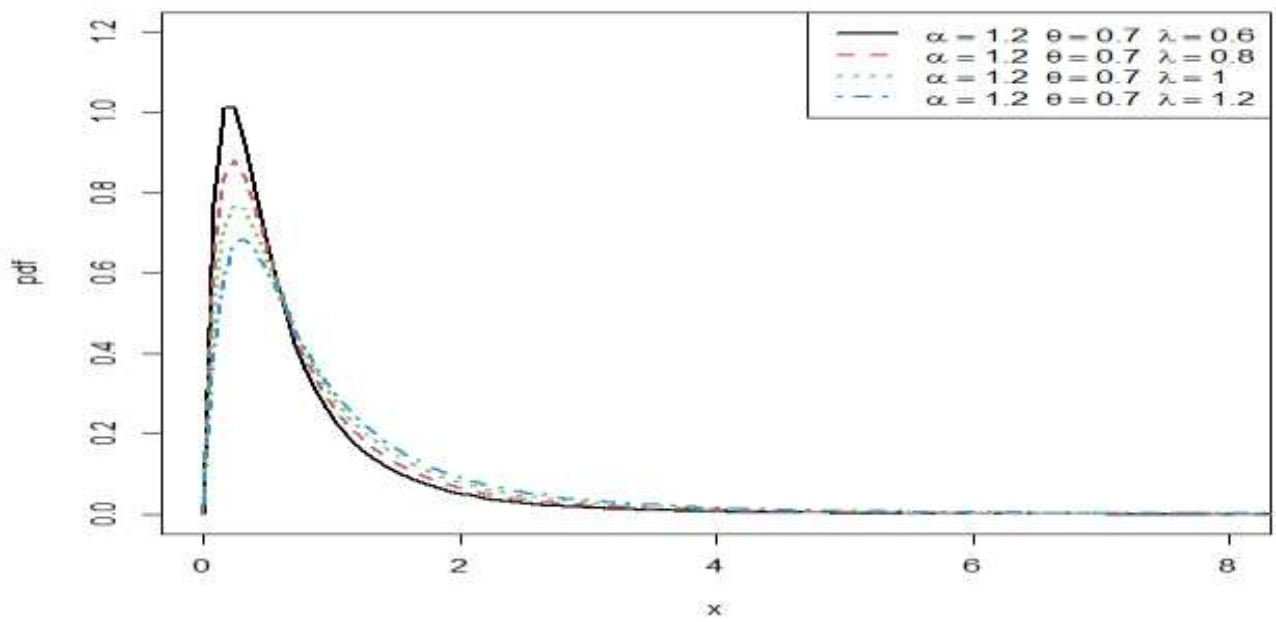


Figure 2: Plot of the pdf of Sine Type II Topp-Leone-Lomax distribution for varying values of λ and fixed values of α and θ

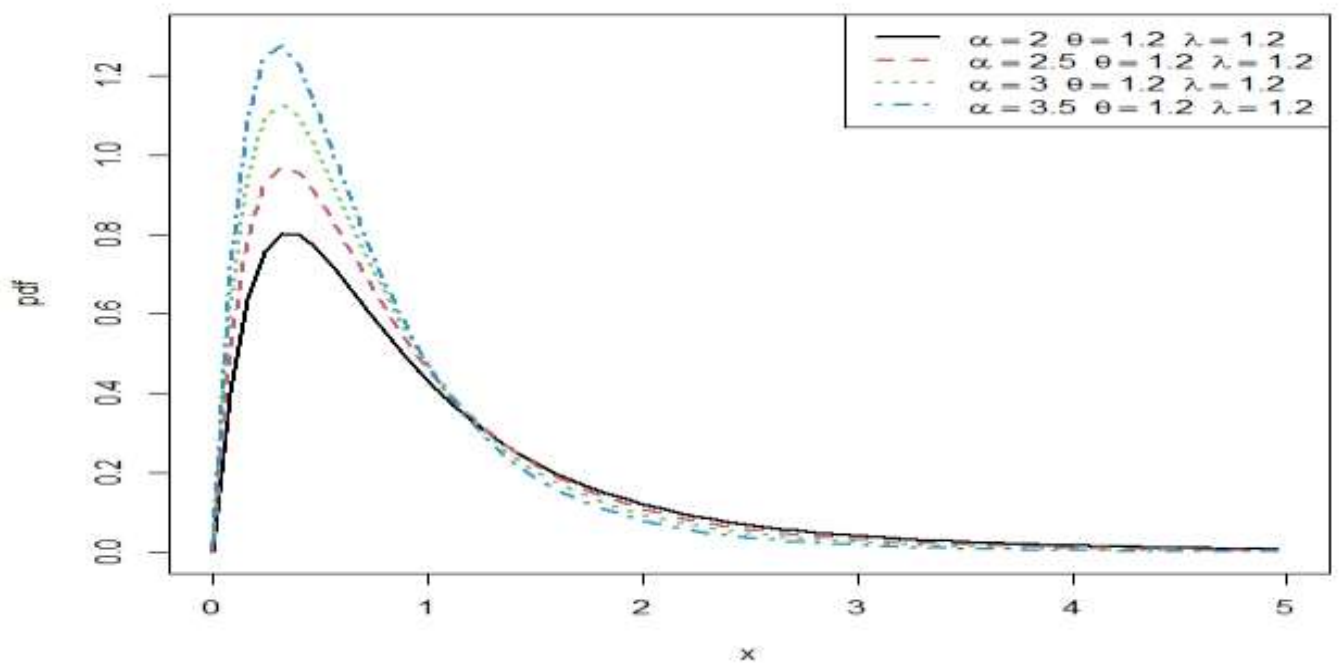


Figure 3: Plot of the pdf of Sine Type II Topp-Leone-Lomax distribution for varying values α and fixed parameter values of λ and θ

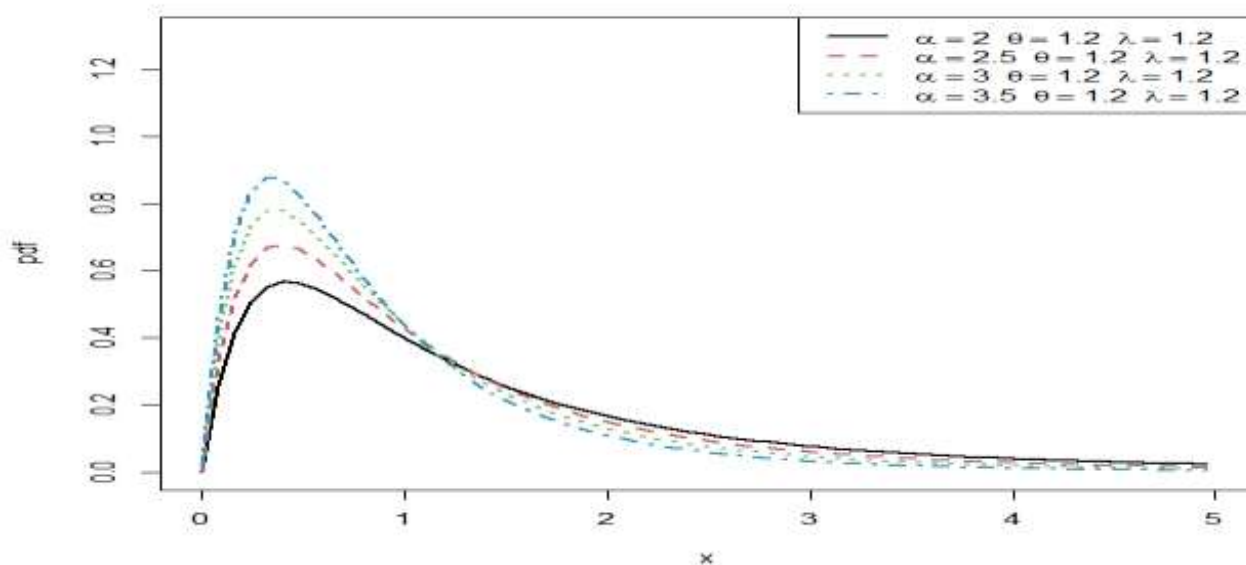


Figure 4: Plot of the pdf of Type II Topp-Leone-Lomax distribution for varying values α and fixed values of λ and θ

Figures 3 and 4 display the plots for two distinct probability distributions: the Sine Type II Topp Leone Lomax distribution and the Type II Topp Leone Lomax distribution. These plots suggest that the probability density function (pdf) plot of the Sine Type II Topp Leone Lomax distribution is more capable of representing datasets with a higher kurtosis compared to the Type II Topp Leone Lomax

distribution at some fixed values of the parameters.

The survival function $S(x; \alpha, \theta, \lambda)$, hazard function $h(x; \alpha, \theta, \lambda)$, reverse hazard function $r(x; \alpha, \theta, \lambda)$, cumulative hazard function $H(x; \alpha, \theta, \lambda)$ and the quantile function $Q(u)$ of the Sine Type II Topp-Leone-Lomax distribution is given below:

$$S(x; \alpha, \theta, \lambda) = 1 - \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right)^\alpha \right] \right\} \tag{21}$$

$$h(x; \alpha, \theta, \lambda) = \frac{\frac{\pi}{2} 2\alpha \frac{\theta}{\lambda} \left(1 + \frac{x}{\lambda} \right)^{-(\theta+1)} \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right) \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^{\alpha-1} \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^\alpha \right\}}{1 - \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^\alpha \right\}} \tag{22}$$

$$r(x; \alpha, \theta, \lambda) = \frac{\frac{\pi}{2} 2\alpha \left[\frac{\theta}{\lambda} \left(1 + \frac{x}{\lambda} \right)^{-(\theta+1)} \right] \left[1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right] \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^{\alpha-1}}{\cot \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^\alpha \right\}} \tag{23}$$

$$H(x; \alpha, \theta, \lambda) = -\ln \left\{ 1 - \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{x}{\lambda} \right)^{-\theta} \right)^2 \right]^\alpha \right\} \right\} \tag{24}$$

$$Q(u) = \lambda \left\{ 1 - \left[1 - \left[1 - \left(\frac{\sin^{-1} u}{\pi/2} \right)^{1/\alpha} \right]^{1/2} \right]^{-1/\theta} - 1 \right\} \tag{25}$$

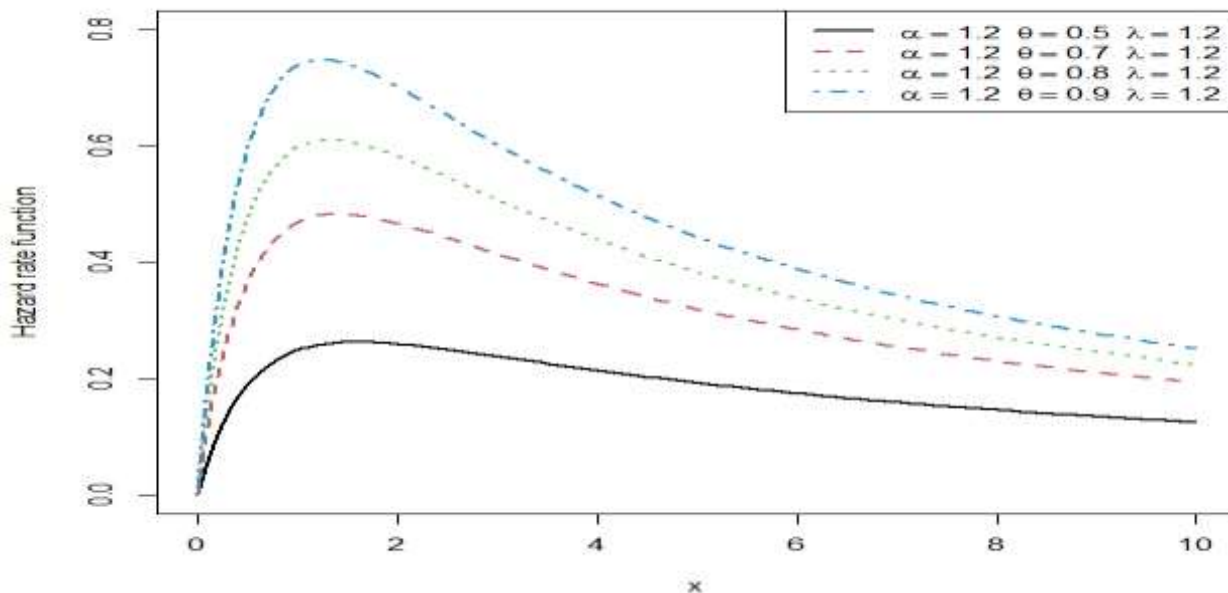


Figure 4: Hazard rate function of the Type II Topp-Leone-Lomax distribution for varying values λ and fixed parameter values of α and θ

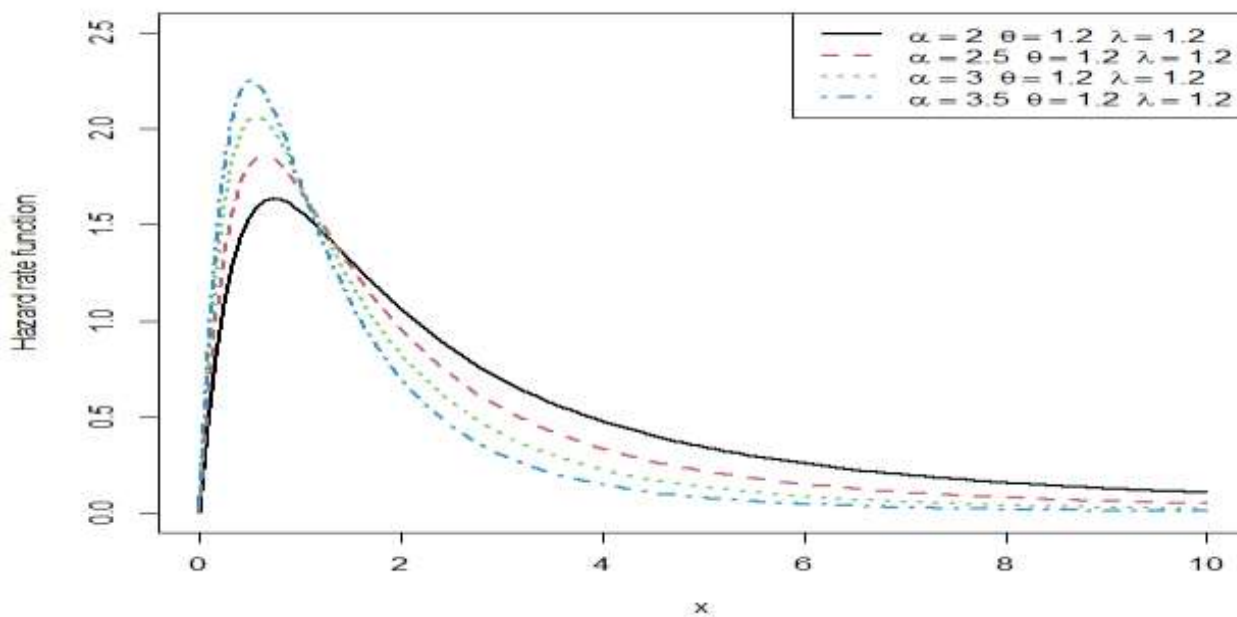


Figure 5: Hazard plot of the Type II Topp-Leone-Lomax distribution for varying values λ and fixed parameter values of α and θ

RESULTS AND DISCUSSION

Assessing the Consistency of the Parameter Estimates

To assess the performance of the newly proposed Sine-Type II Tope Leone-Lomax distribution, a simulation study was conducted using the Monte Carlo Simulation method to compute the mean, bias and mean square error of the estimated parameters from the

maximum likelihood estimates. The Simulated data is produced using the quantile function defined in equation (25) for different sample sizes: $n = 20, 50, 100, 250, 500, 1000$ and replicated 1000 times. For each sample size $\lambda = 2.0, \alpha = 0.5$ and $\theta = 1.1$. Table 1 presents the results of the estimates, bias and root mean square error from the new distribution.

Table 1: Estimate, Bias and RMSE of the new STIITL-Lomax Distribution

N	Properties	$\lambda = 2.0$	$\alpha = 0.5$	$\theta = 1.1$
20	Est.	2.8860	1.2748	1.3502
	Bias	0.8860	0.7748	0.2502
	RMSE	1.9829	1.0582	0.5840
50	Est.	2.4062	1.0907	1.2976
	Bias	0.4062	0.5907	0.1976
	RMSE	0.8354	0.6883	0.3792
100	Est.	2.3147	1.0257	1.2975
	Bias	0.3147	0.5257	0.1975
	RMSE	0.5756	0.5743	0.3232
250	Est.	2.2444	1.0032	1.2805
	Bias	0.2444	0.5032	0.1805
	RMSE	0.3780	0.5317	0.2595
500	Est.	2.2179	0.9896	1.2755
	Bias	0.2179	0.4896	0.1755
	RMSE	0.2902	0.5030	0.2234
1000	Est.	2.2093	0.9888	1.2682
	Bias	0.2093	0.4888	0.1682
	RMSE	0.2491	0.4953	0.1910

The results of the Monte Carlo Simulations are shown in table 1 above. The values of biases and RMSEs tend to zero as shown in Table 1 and the estimates tend to the true parameter values as the sample size increases, indicating that the estimates are efficient and consistent.

Application

Two datasets were considered for illustrative purposes and comparison with the baseline distribution (Lomax) and other extensions of the Lomax distribution such as Type II Tope Leone Lomax distribution, Half Logistic-Lomax distribution developed by Anwar and Zahoor (2018), Weibull Lomax distribution by Tahir *et al.*, (2015), and Gompertz Lomax distribution by (Oguntunde *et al.*, 2017). For

each data set, we estimated the unknown parameters of each distribution by the maximum-likelihood method and also obtained the values of the Akaike information criterion (AIC) for the proposed model and the competitors.

First Dataset

The first data set as listed below represents the COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Al-Marzouki *et al.*, (2020) and Bello *et al.*, (2021): 2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36, 102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.

Second Dataset

The second dataset consists of 100 observations of breaking stress of carbon fibers given by Nicholas and Padgett (2006): 0.920, 0.9280, 0.997, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480,

1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.814, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.3060.

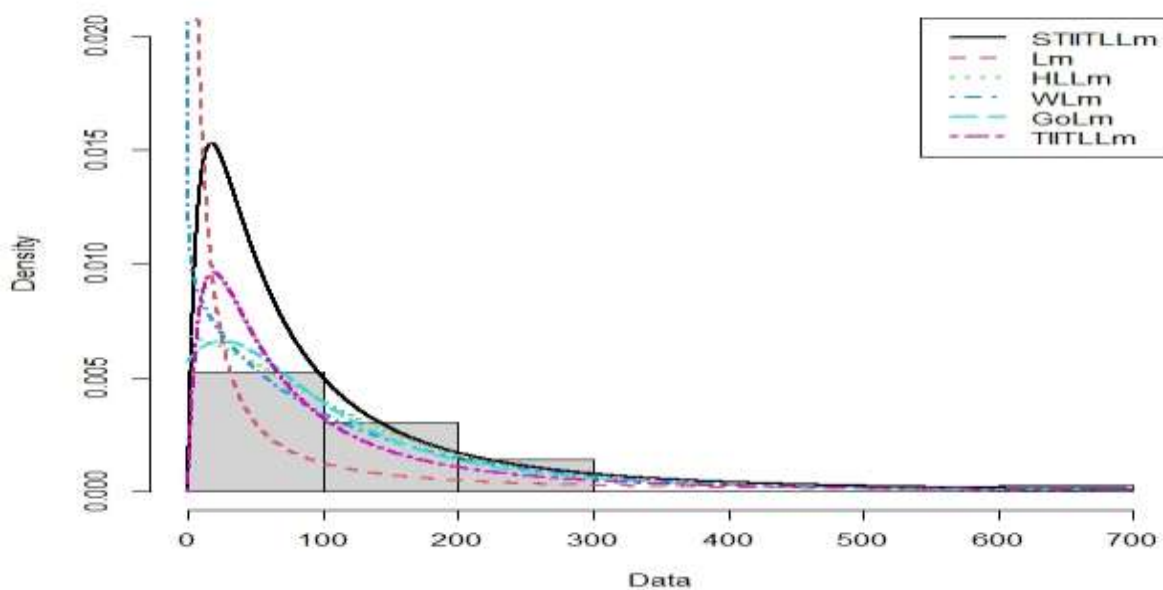


Figure 6: Density plot of the Sine-Type II Topp-Leone-Lomax distribution

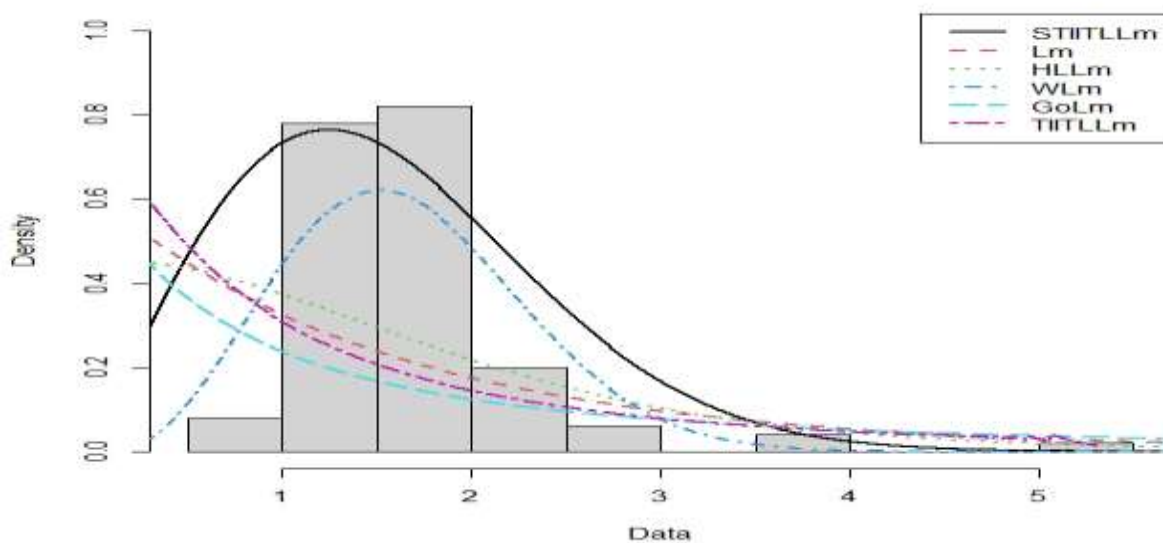


Figure 7: Density plot of the Sine-Type II Topp-Leone-Lomax distribution

Table 2 and table 3 gives the summary statistics of the two data sets such as the mean, the median, the first and third quartile, the minimum and the maximum values.

Table 2: Summary statistics of the two datasets

Data	Minimum	Q_1	Media	Mean	Q_3	Maximum
Dataset I	2.00	32.75	93.00	119.28	183.25	627.0
Dataset II	0.92	1.302	1.544	1.658	1.8140	5.306

Table 3: MLE and AIC of the two data sets

Data set I	α	λ	θ	γ	LL	AIC
STIITL-Lm	5.55080	0.02770	7.4223	–	–197.1501	400.3121
TIITL-Lm	0.18430	24.5957	14.4732	–	–213.8879	433.7758
Lm	0.41228	6.73500	–	–	–224.5627	453.1253
HL-Lm	7.14437	0.00191	–	–	–208.369	420.7379
W-Lm	0.01813	0.84166	1.12087	1.67046	–207.9697	423.9394
Go-Lm	7.96169	2.14265	47.014355	–	–209.7675	425.5349
Dataset II	α	λ	θ	γ	LL	AIC
STIITL-Lm	5.62556	3.98184	1.56784	–	–52.81931	111.6386
TIITL-Lm	34.2579	2.67719	0.06294	–	–167.6184	341.2368
Lm	17.1694	27.7386	–	–	–153.0699	310.1398
HL-Lm	0.05118	0.00178	–	–	–134.2354	272.4707
W-Lm	5.54118	5.62194	0.23070	0.18368	–83.24278	174.4856
Go-Lm	0.03744	0.01936	0.00819	–	–186.4186	378.8372

Table 3 gives the results of the analysis of the two datasets. The results of the analysis of the Sine-Type II Topp-Leone-Lomax was compared with Lomax distribution, Half-Logistic-Lomax distribution, Weibull-Lomax distribution and Gompertz-Lomax. The proposed Sine-Type II Topp-Leone-Lomax distribution has proven to be the better model because it has the least AIC.

The visual examination of the fit presented in Figure 6 and 7 also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution ‘best fits’ both COVID 19 and breaking stress of carbon fibers data sets amongst the variety of competing distributions considered.

CONCLUSION

There has been a growing interest among statisticians and applied researchers in developing flexible lifetime models for the betterment of modelling survival data. In this paper, we introduced a three-parameter Sine-Type II Topp-Leone-Lomax distribution which is obtained by considering a Sine-G family and Type II Topp-Leone-G family of distributions. We study some of its statistical and mathematical properties. Maximum Likelihood Estimation was used in parameter estimation. The usefulness of the new distribution was illustrated via the analysis of two real data sets. We hope that the proposed extended model will attract wider applications in survival analysis.

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