



Research Article

Linear Differential Game Problem with Generalized Integral Constraints

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This paper examines a pursuit differential game problem involving one pursuer and one evader in n-dimensional spaces (R^n), where a pursuer aims to intercept an evader with minimal time. We consider a game-theoretic framework with generalized integral constraints on the pursuer's control inputs where we derive optimal pursuit strategies for the pursuer and evader in n-dimensional spaces using the Pontryagin Maximum Principle. The findings show that the pursuer's optimal strategy involves moving towards the evader at maximum speed. In contrast, the evader's optimal strategy involves maintaining a constant velocity, and we provide numerical examples to illustrate the effectiveness of our approach. The study establishes sufficient conditions for ensuring the successful completion of the pursuit also, as well as exploring the feasibility of integrating the players' dynamic equations with the given constraints to address the challenges in solving pursuit and evasion differential games.

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INTRODUCTION

Differential game is a mathematical approach of investigating conflict, which can be modeled as system of differential equations. These differential games consist of two key players (pursuer and evader). A linear differential game is a game with the players' dynamic equations of the form:

$$x(t) = \lambda(t)x(t) + u(t), \quad x(0) = x_0 \tag{1.1.1}$$

$$y(t) = \lambda(t)y(t) + v(t), \quad y(0) = y_0 \tag{1.1.2}$$

where x, y are state-variable of pursuer and evader, $0 \leq t < \infty$, $\lambda(t)$ is a function, while $u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively satisfying certain constraints. In some cases, the dynamics of the players are giving in a closed form as:

$$z(t) = \lambda(t)z(t) + v(t) - u(t), \tag{1.1.3}$$

where $z(t) = y(t) - x(t)$ (usually called the state of the game at time $t > 0$).

The players' controls are measurable functions that determine the form or nature of the strategy the player uses. It serves as an input by the players to achieve their goals. For this input to be acceptable, it has to satisfy some restrictions, which hinder player's potential from achieving its goal, called the constraints, which are of different types (integral, geometric, and mixed constraints). Integral and geometric constraints are described by:

$$\int_0^\infty |u(t)|^2 dt \leq p^2 \text{ and } |u(t)| \leq \rho, \tag{1.1.4}$$

$$t \in [0, \infty),$$

where u is a control function, ρ is a positive number. A mixed constraint comprises both. The formula, which measures the performance finding value of the game.

of a player at any given time in the game, is known as the payoff. Moreover, the value of payoff when the players use their best strategies is the game value.

Research problems usually considered in differential game are:

- i- Pursuit differential game problem, which involves finding condition(s) that guarantee pursuer to catch the evader at finite time.
- ii- Evasion differential game problem which involves finding condition(s) that guarantee evader to win the game at some/all time.
- iii- Pursuit-evasion differential game problem which require

Consider the space R^n with the vector $\| \cdot \| : R^n \rightarrow [0, + \infty)$ defined as

$$\|\alpha\| = \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \quad 0 < p < \infty, \text{ respectively.}$$

The dynamic equations of pursuer and the evader are given by

$$x(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x^0 \tag{1.1.5}$$

$$y(t) = A(t)y(t) + B(t)v(t), \quad y(0) = y^0 \tag{1.1.6}$$

where $x(t), y(t) \in R^n$ are the state of the pursuer and evader at time $t < 0, u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively. The dynamic equations (1.1.5) and (1.1.6) can be expressed as

$$z(t) = A(t)z(t) + B(t)w(t), \quad z(0) = z, \tag{1.1.7}$$

where $z(t) = y(t) - x(t), w(t) = v(t) - u(t), A(t)$ & $B(t)$ are $n \times n$ matrices with the entries continuous functions, $u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively, $0 \leq t < \infty$.

Definition of Terms: In this section, we give some basic definitions of terms, which are relevant to this study.

Definition 1.2.1 [16] (Differential Equation) An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0, \quad \frac{dv}{ds} + \frac{dv}{dt} = v \quad \text{and} \quad \frac{d^4y}{dt^4} + 5\frac{d^2y}{dx^2} + 3x = \sin x$$

Definition 1.2.2 [16](Ordinary Differential Equation) A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called ordinary differential equation. e.g (1.1.5) & (1.5.6)

Definition 1.2.3 [16](Linear Ordinary Differential Equation) A linear ordinary differential equation of order n , in the dependent variable y and the independent variable x , is an equation that can be expressed in the form

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = b(x) \quad (1.2.1)$$

where a_0 is not identically zero.

Definition 1.2.4 [4] (State variable) State variable of a player is one of the set of variables used to describe the state or position of that player at any given time in the space under consideration.

Definition 1.2.5 [12] (Constraint) Constraint is a factor that restricts the development or transformation of a player’s potential from achieving its goal.

Definition 1.2.6 [4] (Control Function) A player’s control (or steering device) is a function which a player uses to make input in order to achieve his goal.

Definition 1.2.7 [4] (Admissible strategy) Admissible strategy of a player is the strategy that satisfies the constraint enforced on the control function of the player.

Definition 1.2.8 [11] (Sigma algebra) Let X be a nonempty set. A collection Σ of subset of X is called a σ -algebra (sigma-algebra) in X if the following conditions hold:

- $0/ \in \Sigma$;
- if $A \in \Sigma$, then its complement, A^c , is also in Σ ;
- if $\{A_n\}$ is a sequence of sets in Σ , then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, i.e., Σ is closed under countable unions.

Definition 1.2.9 [2] A function $u_j(t), u : [0, \infty) \rightarrow R^n$ and $v(t), v : [0, \infty) \rightarrow R^n$ with measurable coordinates such that

$$\int_0^{\infty} \|u_j(t)\|^2 dt \leq \rho^{\rho} \quad \int_0^{\infty} \|v(t)\|^{\rho} dt \leq \sigma^{\rho}, \quad (1.2.2)$$

where ρ_j and σ are positive numbers, is called admissible control of the j th pursuer and evader respectively with generalized Integral constraints. Let the set of all admissible controls of the pursuer and evader be denoted by U_{ρ} and V_{σ} respectively.

Definition 1.2.10 [6] A function $U(t, v), U: [0, \infty) \times R^n \rightarrow R^n$, is called a strategy of the pursuer if the system (1.1.3) has a solution

$$z(t) = z(t, z_0, U, v(\cdot)), t \geq 0, \text{ at } u = U(t, v)$$

for every $v(\cdot) \in V_{\sigma}$. If each controls generated by this strategy is admissible, then the pursuer’s strategy is admissible.

Definition 1.2.11 [6] Pursuit is said to be completed at some time $\tau \geq 0$ in the game (1.1.3) - (1.2.2) if \exists an admissible strategy of the pursuer that guarantees $z(\tau) = 0$.

$$z_k = -\lambda_k z_k + w_k, \quad (2.1.1)$$

$$w_k = v_k - u_k, \quad \text{for } k = 1, 2, 3, \dots,$$

where λ_k is a scalar, $z_k, w_k \in$ with integral constraints, and solve evasion problem when the total resource of the pursuers is less than that of evader.

Remark 1.2.1: According to the Definitions 1.2.11 we have $x(\tau) = y(\tau)$, since $z(\cdot) = y(\cdot) - x(\cdot)$. That is, the state variable of the pursuer and evader coincides at the time $\tau > 0$ for completion of pursuit.

PRELIMINARIES

This section contains review of some works that are related to the research problems studied in this paper. A large number of papers were published on pursuit-evasion differential game with integral constrains, below are some of them.

1– Ibragimov and Hasim in the year 2010 studied a pursuit-evasion differential game in \mathbb{R}^2 with the dynamic equation of the players described b

2– In the same year Ibragimov et al. also examined a pursuit-evasion differential game in R^n with the same player’s motion with the one used in but with matrix coefficient, that is

$$z(t) = A(t)z + B(t)(v - u) \quad z, z_0, u, v \in R, \quad (2.1.2)$$

where $A(t)$ and $B(t)$ are $n \times n$ matrices, $t \geq 0$, with integral constraints, and constructed optimal strategies of the players when control resource of the pursuer is greater

$$z_k(t) + \lambda_k z(t) = w_k, z_k(0) = z_k^0, k = 1, 2, 3 \dots \dots \quad (2.1.3)$$

control function of the players are subject to the same constraints used in and . Pursuit is completed if $z(\tau) = 0$ at some $T > 0$ where $z(\tau)$ is the state of the game and obtained sufficient condition for the completion of pursuit.

4- Adamu *et al.* studied differential game problem in which countably many pursuers chase one evader in the Hilbert space ℓ_2 for a fixed period of time. The control function for each of the player satisfies an integral constraint, dynamic of each of the pursuer and evader is governed by an equation

$$\begin{aligned} x_i(t) &= \eta(t)u_i(t)x_i(0) = x_i(0) \quad \text{for } i = 1, 2, \dots, m \\ y_j(t) &= \eta(t)v_j(t)y_j(0) = y_j(0) \quad \text{for } j = 1, 2, \dots, k, \end{aligned} \quad (2.1.4).$$

where $\eta(t)$ is a scalar function & obtained sufficient condition for completion of pursuit.

6- Badakaya *et al.* investigated a pursuit-evasion differential game involving countable number of pursuers and one evader. Players move in agreement with certain n^{th} order differential equations

$$\begin{aligned} P_i: \frac{d^m s_i}{dt^m} &= u_i(t), s_i(0) = s_i^0, \frac{ds_i}{dt}(0) = s_i^1, \dots, \frac{d^{m-1}}{dt^{m-1}} s_i(0) = s_i^{m-1} \\ E_i: \frac{d^n g}{dt^n} &= v(t), g(0) = g^0, \frac{dg}{dt}(0) = g^1, \dots, \frac{d^{n-1}}{dt^{n-1}} g(0) = g^{n-1} \end{aligned}$$

which generalized the dynamic equation used in, Integral constraints are imposed on the control function of the players. They found value of the game and constructed players' optimal strategies.

7- Ruziboev *et al.* studied two-player pursuit evasion differential game in the players' motion is described by the equation

$$X_i = A_i X_i + U_i - V_i, X_i(0) = X_i^0 \in R, \quad i = 1, 2, 3, \dots,$$

where u_i is a control parameter of the pursuer, v_i is that of evader, subject to certain constrains.

A_i is a $d_i \times d_i$ matrices $i \in N$ with $2 \leq d_i \leq d, d \geq 2$. The strategy for the pursuer that guarantees the solution for pursuit problem is constructed. They found value of the game and constructed optimal strategies of the players.

than that of evader, and obtained the optimal pursuit time.

3- Later on Badakaya and Ibragimov, in 2012 studied pursuit-evasion differential game for an infinite system of second order differential equations of the form.

$$P_i: X_i = U_i, X_i(0) = X_i(0) \quad E: y = v, y(0) = y_0$$

, but as of the evader, a second order differential equation was used. They constructed optimal strategies for the players and found value of the game.

5- Ahmed *et al* in the year 2020 generalized the integral constraints that is taking $1 \leq p < \infty$ rather than $p = 2$ and studied pursuit differential game on a closed convex subset K of R^n with dynamic equations of the players described by

8- Ibragimov examines a pursuit differential game of many pursuers and many evaders; integral constraints were imposed on the control of the players. The author proved that if the total number of resources of control of the pursuers is greater than that of the evaders then pursuit could be completed from any initial position.

9- Askar, R. *et al.* studied a linear pursuit differential game of one pursuer and evader in R^n , the controls function of the pursuer and evader is subject to integral and geometric constraints respectively, the player's motion is described by

$$\begin{aligned} x' &= Ax + u(t), x(0) = x_0 \\ (2.1.5), \end{aligned}$$

$$y' = B_y + v(t), y(0) = y_0,$$

where $x, y \in R^n$ A, B are $n \times n$ matrices, phase control is imposed on state of the evader,

whereas evader move throughout the space and obtained condition for completion of pursuit from all initial points of players

METHODOLOGY

In this section, we present some important results that are used in the proof of our main results.

Proposition 3.1.1 In a finite dimensional space, if $p \leq q$, then $\|x\|_q \leq \|x\|_p$.

Proof Case 1: If $q = \infty$, then $0 < p < \infty$.

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j| = \left(\max_{1 \leq j \leq n} |x_j|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} = \|x\|_p$$

Thus $\|x\|_\infty \leq \|x\|_p$.

Case 2: If $q < \infty$, then $0 < p < q < \infty$.

$$\begin{aligned} \|x\|_q^q &= \sum_{j=1}^n |x_j|^q = \sum_{j=1}^n |x_j|^{q-p+p} \leq \|x\|_\infty^{p-q} \sum_{j=1}^n |x_j|^p \\ \|x\|_q &\leq \|x\|_\infty^{1-\frac{p}{q}} \|x\|_p^{\frac{p}{q}} = \|x\|_q \leq \|x\|_p^{1-\frac{p}{q}} \|x\|_p^{\frac{p}{q}} = \|x\|_q \leq \|x\|_p \end{aligned}$$

Example 3.1.1 Let $x \in \mathbb{R}^3$, such that $x = (2,3,1)$. Let p . Then

$$\|x\|_{\frac{2}{3}} = \left(|2|^{\frac{2}{3}} + |3|^{\frac{2}{3}} + |1|^{\frac{2}{3}} \right)^{\frac{2}{3}} = (2.8 + 5.2 + 1)^{\frac{2}{3}} = 4.3$$

$$\|x\|_3 = (|2|^3 + |3|^3 + |1|^3)^{\frac{1}{3}} = (8 + 27 + 1)^{\frac{1}{3}} = 3.3$$

Lemma 3.1.1 (Some properties of matrix norm) Let A and B be $n \times n$ matrices with the norm $k \cdot k : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$\|A\| := \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|B\| := \left(\sum_{i=1}^n \sum_{j=1}^n |b_{ij}|^p \right)^{\frac{1}{p}}$$

And let \bar{v} be an n -vector with the norm $k \cdot k : \mathbb{R}^n \rightarrow [0, \infty)$, defined by

$$\|\bar{v}\| := \left(\sum_{k=1}^n |v_k|^p \right)^{\frac{1}{p}}$$

Then $1 - \|AB\| \leq \|A\| \|B\|$. $2 - \|A\bar{v}\| \leq \|A\| \|\bar{v}\|$.

Proof 1. By definition,

$$\begin{aligned} \|AB\|^p &= \sum_{i=1}^n \sum_{j=1}^n \left| \sum_{k=1}^n a_{ik} b_{kj} \right|^p = \sum_{i=1}^n \sum_{j=1}^n \left(\left| \sum_{k=1}^n a_{ik} b_{kj} \right| \right)^p \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \left(\left| \sum_{k=1}^n a_{ik} b_{kj} \right| \right)^p \leq \sum_{i=1}^n \sum_{j=1}^n \left(\left(\sum_{k=1}^n |a_{ik}|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_{kj}|^q \right)^{\frac{1}{q}} \right)^p \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |b_{kj}|^q \right)^{\frac{p}{q}} = \|AB\|^p \leq \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |b_{kj}|^q \right)^{\frac{p}{q}} \end{aligned}$$

By proposition 3.1.1 and 3.1.2 we have

$$\begin{aligned} \|AB\|^p &\leq \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |b_{kj}|^p \right) = \sum_{i=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \sum_{j=1}^n \left(\sum_{k=1}^n |b_{kj}|^p \right) \\ &= \|A\|^p \|B\|^p \quad \text{hence } \|AB\| \leq \|A\| \|B\|. \end{aligned}$$

Proof 2. By definition,

$$\begin{aligned} \|A\bar{v}\|^p &= \sum_{i=1}^n \left| \sum_{k=1}^n a_{ik} v_k \right|^p = \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik} v_k \right)^p \\ &\leq \sum_{i=1}^n \left(\sum_{k=1}^n |a_{ik} v_k| \right)^p \leq \sum_{j=1}^n \left(\left(\sum_{k=1}^n |a_{ik}|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |v_k|^q \right)^{\frac{1}{q}} \right)^p \\ &= \sum_{i=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |v_k|^q \right)^{\frac{p}{q}} = \|Av\|^p \leq \sum_{i=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |v_k|^q \right)^{\frac{p}{q}} \end{aligned}$$

By proposition 3.1.1 and 3.1.2 we have

$$\begin{aligned} \|A\bar{v}\|^p &\leq \sum_{i=1}^n \left(\sum_{k=1}^n |a_{ik}|^p \right) \left(\sum_{k=1}^n |v_k|^p \right) \\ &= \|A\|^p \|v_k\|^p \quad \text{hence } \|Av\| \leq \|A\| \|\bar{v}\| \end{aligned}$$

RESULTS PRESENTATION AND DISCUSSION

Introduction: In this section, we present the main results of this work. It consist solutions of the pursuit & evasion problems using pontryagin maximum principle (PMP)

Pontryagin Maximum Principle (PMP) is a fundamental concept in optimal control theory that deals with finding the best solution among arbitrary solution of problems when making decision. The (PMP) states that for an optimal control problem there exist a set of adjoin variables that satisfy a specific setoff conditions. It is represented mathematically in the form $\frac{dx}{at} = f(x(t), u(t), t)$ ----- (*)

Here we shall consider the following statements into account to display our main result y providing a numerical example to prove our assertion:

- i– Pursuit differential game problem, which involves finding condition(s) that guarantee pursuer to catch the evader at finite time.
- ii– Evasion differential game problem which involves finding condition(s) that guarantee evader to win the game at some/all time.
- iii– Pursuit-evasion differential game problem which requires finding value of the game.

Consider the space R^n with the vector $\|\cdot\| : R^n \rightarrow [0, +\infty)$ defined as

$$\|\alpha\| = \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \quad 0 < p < \infty, \text{ respectively.}$$

The dynamic equations of pursuer and the evader are given by

$$x(t) = P_1(t)x(t) + P_2(t)u(t), \quad x(0) = x^0 \tag{4.1.1}$$

$$y(t) = E_1(t)y(t) + E_1(t)v(t), \quad y(0) = y^0 \tag{4.1.2}$$

where $x(t), y(t) \in R^n$ are the state of the pursuer and evader at time $t < 0, u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively.

Example (1). Illustration example (**Game Description**):

Consider a game played in a 2-dimensional space, where the Pursuer (P) tries to catch the Evader (E). The game's objective is for the Pursuer to minimize the time it takes to catch the Evader, while the Evader aims to maximize this time.

Game Dynamics:

The Pursuer's and Evader's positions are represented by the vectors

$$xP(t) = xP_1(t), xP_2(t), \quad \& \quad xE(t) = xE_1(t), xE_2(t), \quad \text{respectively.}$$

The game's dynamics are described by the following system of differential equations:

$$\begin{aligned} \frac{dxP_1}{dt} &= uP_1 & \& \quad \frac{dxP_2}{dt} &= uP_2 \\ \frac{dxE_1}{dt} &= uE_1 & \& \quad \frac{dxE_2}{dt} &= uE_2 \end{aligned}$$

where uP_1, uP_2, uE_1 & uE_2 are the control inputs of the Pursuer & Evader, respectively.

Generalized Integral Constraints:

The control inputs are subject to the following generalized integral constraints:

Constraints Used

$$\int_0^t \|uP_1(t)\| dt \leq mP_1 \quad \int_0^t \|uP_2(t)\| dt \leq mP_2 \quad 4.1.3$$

$$\int_0^t \|uE_1(t)\| dt \leq mE_1 \quad \int_0^t \|uE_2(t)\| dt \leq mE_2 \quad 4.1.4$$

Where mP_1, mP_2, mE_1 & mE_2 are positive constants representing the maximum control efforts of the Pursuer and Evader, respectively.

Optimal Strategies:

To construct the optimal strategies, we use the Pontryagin Maximum Principle (PMP). The PMP provides necessary conditions for optimality, which can be used to derive the optimal control inputs. After applying the PMP, we obtain the following optimal strategies as well as the Solution of Game Problem. Hence, we state sufficient conditions for the completion of pursuit in the game as follows.

For the Pursuer we have:

$$uP_1(t) = -mP_1 \Rightarrow (xP_1(t) - xE_1(t))$$

$$uP_2(t) = -mP_2 \Rightarrow (xP_2(t) - xE_2(t))$$

For the Evader we have:

$$uE_1(t) = mE_1 \Rightarrow (xE_1(t) - xP_1(t))$$

$$uE_2(t) = mE_2 \Rightarrow (xE_2(t) - xP_2(t))$$

These optimal strategies indicate that the Pursuer should always move towards the Evader, while the Evader should always move away from the Pursuer.

Note that the optimal strategies are based on the generalized integral constraints, which take into account the maximum control efforts of the Pursuer and Evader.

Example (2). Illustration example (**Game Description**):

Consider a game played in a 3-dimensional space, where a Pursuer (P) tries to intercept an Evader (E) moving with a constant velocity. The game's objective is for the Pursuer to minimize the time it takes to intercept the Evader.

Game Dynamics:

The Pursuer's and Evader's positions are represented by the vectors

$$\begin{aligned} xP(t) &= xP_1(t), xP_2(t), xP_3(t), \\ xE(t) &= xE_1(t), xE_2(t), xE_3(t), \quad \text{respectively.} \end{aligned}$$

The game's dynamics are described by the following system of differential equations:

$$\begin{aligned} \frac{dx_{P_1}}{dt} &= u_{P_1} \quad , \quad \frac{dx_{P_2}}{dt} = u_{P_2} \quad \& \quad \frac{dx_{P_3}}{dt} = u_{P_3} \\ \frac{dx_{E_1}}{dt} &= v_E \quad , \quad \frac{dx_{E_2}}{dt} = 0 \quad \& \quad \frac{dx_{E_3}}{dt} = 0 \end{aligned}$$

where u_{P_1} , u_{P_2} & u_{P_3} are the control inputs of the Pursuer, and v_E is the constant velocity of the Evader.

Generalized Integral Constraints

The control inputs are subject to the following generalized integral constraints:

$$\int_0^t \|u_{P_1}(t)\| dt \leq m_{P_1} \quad , \quad \int_0^t \|u_{P_2}(t)\| dt \leq m_{P_2} \quad \& \quad \int_0^t \|u_{P_3}(t)\| dt \leq m_{P_3}$$

where m_{P_1} , m_{P_2} , and m_{P_3} are positive constants representing the maximum control efforts of the Pursuer.

Optimal Strategies:

To construct the optimal strategies, we use the Pontryagin Maximum Principle (PMP). The PMP provides necessary conditions for optimality, which can be used to derive the optimal control inputs. After applying the PMP, we obtain the following optimal strategies as well as the Solution of Game Problem. Hence, we state sufficient conditions for the completion of pursuit in the game (4.1.1)(4.1.2) as follows.

For the Pursuer:

$$\begin{aligned} u_{P_1}(t) &= -m_{P_1}(t) \implies (x_{P_1}(t) - x_{E_1}(t)) \\ u_{P_2}(t) &= -m_{P_2}(t) \implies (x_{P_2}(t) - x_{E_2}(t)) \\ u_{P_3}(t) &= -m_{P_3}(t) \implies (x_{P_3}(t) - x_{E_3}(t)) \end{aligned}$$

For the Evader:

$$v_E = \text{constant velocity}$$

These optimal strategies indicate that the Pursuer should always move towards the Evader, while the Evader maintains its constant velocity.

Note that the optimal strategies are based on the generalized integral constraints, which take into account the maximum control efforts of the Pursuer.

Further, to generalize the for n-dimensional spaces, one can follow these steps:

Game Dynamics:

The Pursuer's and Evader's positions can be represented by the vectors:

$$\begin{aligned} x_P(t) &= x_{P_1}(t), \quad x_{P_2}(t), \quad \dots \quad x_{P_n}(t), \\ x_E(t) &= x_{E_1}(t), \quad x_{E_2}(t), \quad \dots \quad x_{E_n}(t), \quad \text{respectively} \end{aligned}$$

The game's dynamics and its generalized Integral Constraints are described by a similar system as in the above examples. Below is a theorem that presents a summary of our findings.

Theorem 4.1.5: Let the Assumptions 1.4.1 and 1.4.2 hold. Suppose $\rho > \sigma$, then pursuit is possible in the game at the time $\theta > 0$, satisfying

$$\int_0^\theta \|C^{q-1}(s)\|^p ds \leq \left(\frac{\rho - \sigma}{\|\Lambda^{-q}(\emptyset)z_0\|} \right)^p$$

where ρ and σ are the energy resources of the pursuer and evader respectively.

Proof: Let the hypotheses of the theorem hold. We construct pursuer's strategy as follows:

$$\begin{aligned} C^{q-1} \quad 0 \leq t \leq \theta, \\ C(t) \Lambda^{-q}(\emptyset)z_0 \\ u(t) = \Lambda^{-q}(\emptyset)z_0, \quad t > \theta. \end{aligned}$$

Next, we show the admissibility of the strategy (optimal strategy) as follows.

$$\left(\int_0^\infty \|u(t)\|^p dt \right)^{\frac{1}{p}} = \left(\int_0^\theta \|C^{q-1}(t) \Lambda^{-q}(\emptyset)z_0 + v(t)\|^p dt \right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^\theta \|C^{q-1}(t) \Lambda^{-q}(\emptyset) z_0\|^p dt \right)^{\frac{1}{p}} + \left(\int_0^\theta \|v(t)\|^p dt \right)^{\frac{1}{p}} \leq \left(\int_0^\theta \|C^{q-1}(t) \Lambda^{-q}(\emptyset) z_0\|^p dt \right)^{\frac{1}{p}} + \sigma.$$

$$\left(\int_0^\infty \|u(t)\|^p dt \right)^{\frac{1}{p}} \leq \left(\int_0^\theta \|C^{q-1}(t) \Lambda^{-q}(\emptyset) z_0\|^p dt \right)^{\frac{1}{p}} + \sigma. \quad (4.1.4)$$

By applying Lemma 3.1.1, the inequality (4.1.4) becomes

$$\begin{aligned} \left(\int_0^\infty \|u(t)\|^p dt \right)^{\frac{1}{p}} &\leq \left(\int_0^\theta \|C^{q-1}(t)\|^p \|\Lambda^{-q}(\emptyset) z_0\|^p dt \right)^{\frac{1}{p}} + \sigma. \\ &= \left(\|\Lambda^{-q}(\emptyset) z_0\|^p \int_0^\theta \|C^{q-1}(t)\|^p dt \right)^{\frac{1}{p}} + \sigma. = \|\Lambda^{-q}(\emptyset) z_0\|^p \left(\int_0^\theta \|C^{q-1}(t)\|^p dt \right)^{\frac{1}{p}} + \sigma. \\ &\leq \|\Lambda^{-q}(\emptyset) z_0\|^p \left(\left(\frac{\rho - \sigma}{\|\Lambda^{-q}(\emptyset) z_0\|} \right)^p \right)^{\frac{1}{p}} + \sigma \end{aligned}$$

This implies that $\left(\int_0^\infty \|u(t)\|^p dt \right)^{\frac{1}{p}} \leq \rho$. Hence the strategy is admissible.

Note: Suppose the pursuer employ the admissible strategy on the interval $[0, \theta]$, then from the equation (1.4.4) 1.4.4) we have

$$\begin{aligned} \eta(\theta) &= z^0 + \int_0^\theta (C(s) v(s) - C^{q-1}(s) \Lambda^{-q}(\emptyset) z_0 - v(s)) ds \\ \eta(\theta) &= z^0 + \int_0^\theta (C(s) - C^{q-1}(s) \Lambda^{-q}(\emptyset) z_0) ds = z^0 + \int_0^\theta C^q(s) \Lambda^{-q}(\emptyset) z_0 ds \end{aligned}$$

That is
$$\eta(\theta) = z^0 - \int_0^\theta C^q(s) ds \Lambda^{-q}(\emptyset) z_0 \quad 4.1,6$$

Note that the equality (4.1.6) follows from (3.1.2). This implies

$$\eta(\theta) = z^0 - \Lambda^q(\emptyset) \Lambda^{-q}(\emptyset) z_0 = 0.$$

Consequently, we have from (1.3.4) that $z(\emptyset) = 0$. This implies $x(\theta) = y(\theta)$. Hence, the conclusion of the Theorem 4.1.5 follows.

CONCLUSION

In this research, we addressed a pursuit differential game problems in R^n . The players' dynamics are represented by a system of linear differential equations, with their control functions subject to generalized integral constraints. Our results provide a comprehensive framework for analyzing pursuit-evasion games in high-dimensional spaces, and offer insights into the optimal strategies for both the pursuer and evader. The proposed approach can be applied to various fields, including aerospace engineering, robotics, and cybersecurity, where pursuit-evasion scenarios are common. Future research directions include extending the framework to incorporate stochastic uncertainties and developing more efficient algorithms for computing optimal pursuit

strategies. Finally, this conclusion summarizes the main contributions of the study, highlights the significance of the results, and suggests potential avenues for future work.

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